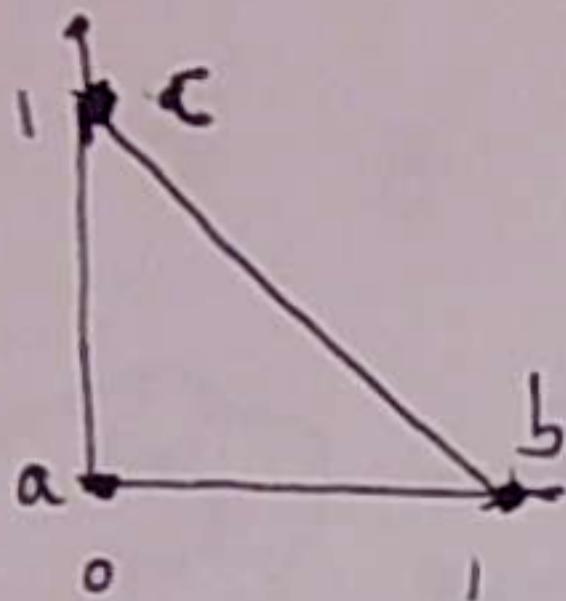


Application to dimension theory



$T \subset \mathbb{R}^2$ closed triangle ("simplex").

$\exists \varepsilon > 0$ s.t. given any finite cover of T by open sets w/ $\text{diam} < \varepsilon$, some pt of T belongs to at least 3 sets of the cover.

(This implies $\text{cov.dim}(T) \not\leq 1$) know $\text{w.v.dim} \leq 2$

Rk true for std simplex in \mathbb{R}^n .

Idea ~~if~~ $\varepsilon = \frac{1}{2}$. Then any set w/ $\text{diam} < \frac{1}{2}$ intersects at most 2 edges.

$$\Delta = \{U_1, \dots, U_n\} \quad \text{diam}(U_i) < \frac{1}{2}.$$

Assign to each U_i a vertex v_i of ∂T

U_i intersects 2 edges $\rightarrow v_i = \text{common vertex}$

U_i intersects 1 edge \rightarrow pick the one of the vertices.

U_i intersects 0 edges \rightarrow pick any vertex

Let (φ_i) be a subordinate (cont.) partition of 1.

$$\text{let } f: T \rightarrow \mathbb{R}^2 \quad f(x) = \sum_{i=1}^n \varphi_i(x) v_i \quad (\text{cont})$$

check [in fact since no 3 U_i 's intersect f maps T to ∂T
also maps each edge to itself. $\rightarrow f|_{\partial T} \simeq \text{id}_{\partial T}$ (straight line
contr. \rightarrow)]

Last time

$$A \subset X \quad (\text{closed}). \quad x_0 \in A$$

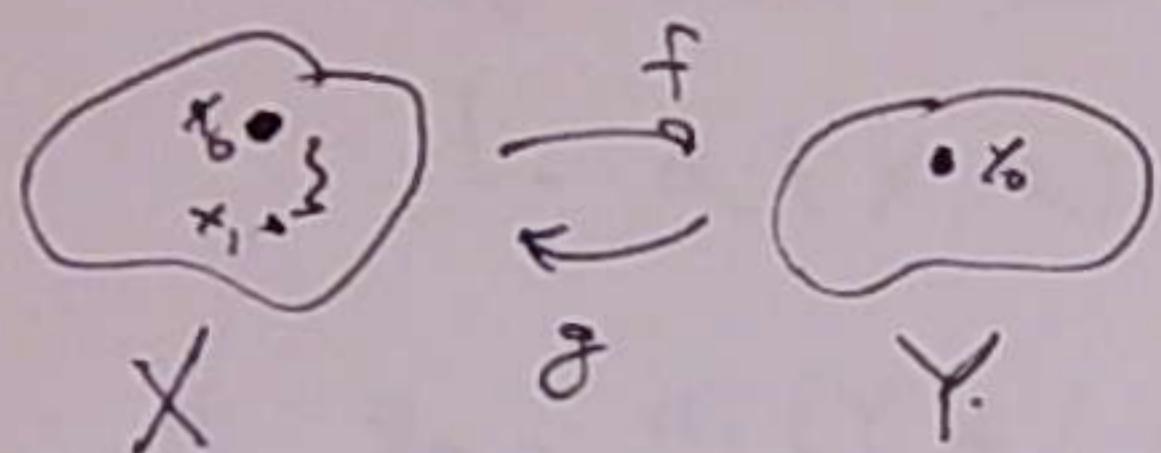
A retract if $\exists r: X \rightarrow A$ cont., $r|_A = \text{id}_A$ $j: A \hookrightarrow X$
 $(\Rightarrow j_*: \pi_1(A, x_0) \rightarrow \pi_1(X, x_0) \text{ is injective})$

A def. retract If $\exists \{r_s\}_{s \in [0,1]}$ homotopy from $r_0 = \text{id}_X$ ($\Leftrightarrow j_*$ is iso)
 $\rightarrow r_1: X \xrightarrow{\text{retract}} A$ ($r_s|_A = \text{id}_A \forall s$)

Def $f: (X, x_0) \xrightarrow{\text{cont.}} (Y, y_0)$ is a homotopy equivalence

If $\exists g: (Y, y_0) \rightarrow (X, x_0)$ s.t. $g \circ f \simeq \text{id}_X$

$$f \circ g \simeq \text{id}_Y$$



Theorem $f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is iso

Recall X connected, $\alpha: x_0 \rightsquigarrow x_1$

then $\hat{\alpha}: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$. \cup iso

$$\hat{\alpha}([f]) = [(\overline{\alpha} * f) * \alpha] \in \pi_1(X, x_1).$$

$$f \in \Omega_{x_0}(X) \quad \Omega_{x_1}(X)$$

Lemma $\exists \alpha: x_0 \rightsquigarrow x_1$ s.t. $g_* \circ f_* = \hat{\alpha}$

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1)$$

given the lemma

$\hat{\alpha}$ is iso, so g_* is onto

given $[\gamma] \in \pi_1(X, x_1)$, let $[\beta] \in \pi_1(X, x_0)$ s.t. $\hat{\alpha}[\beta] = [\gamma]$

Consider $f_*[\beta] \in \pi_1(Y, y_0)$

$$\text{By } g_*(f_*[\beta]) = (g_* \circ f_*)[\beta] = \hat{\alpha}[\beta] = [\gamma]$$

(since g_* onto). Also f_* is injective

$$f_*[\beta] = [e_{y_0}] \rightarrow g_*[e_{x_0}] = (g_* \circ f_*[\beta] = \hat{\alpha}[\beta] =$$

$$[e_{x_1}])$$

$$\hat{\alpha} \text{ is iso} \quad \forall \beta \quad \hat{\alpha}[\beta] = \beta \quad \text{if } \beta = e_{x_0}$$

Likewise

$$f_* \circ g_* : \pi_1(Y, y) \longrightarrow \pi_1(Y, f(x)) \text{ is } \underline{\cong}$$

so f_* is onto and g_* is injective

$$\hookrightarrow \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$$

$$\text{so } g_* : \pi_1(Y, y) \longrightarrow \pi_1(X, x_0) \text{ is } \underline{\cong}$$

Lemma : $g_* \circ f_* = \hat{\alpha}$, so $f_* = g_*^{-1} \circ \hat{\alpha} : \pi_1(X, x_0) \rightarrow \pi_1(Y, y)$
 is $\underline{\cong}$ (composition of iso)

Pf of lemma

In gen'l if $f, g : X \longrightarrow Y$ $\begin{pmatrix} f(x_0) = y_0 \\ g(x_0) = y_1 \end{pmatrix}$
 $f \simeq g$ via $h_s : X \longrightarrow Y$ ($s \in [0,1]$)

then if $\alpha(s) = h_s(x_0)$ $\alpha : y_0 \rightsquigarrow y_1$

we have

$$g_* = \hat{\alpha} \circ f_*$$

want

$$[\beta] \in \pi_1(X, x_0) \quad \beta \in \Omega_{x_0}(X)$$

$$g_* [\beta] = [g \circ \beta] \in \pi_1(Y, y_1).$$

$$\hookrightarrow \Omega_{y_1}(Y)$$

$$f_* [\beta] = [\underbrace{f \circ \beta}_{\Omega_{y_0}(Y)}] \text{ in } \pi_1(Y, y_0)$$

$$\hat{\alpha} (f_* [\beta]) = [\underbrace{(\hat{\alpha} * (f \circ \beta)) * \alpha}_{\Omega_{y_0}(Y)}]$$

so the claim is the loops at $y_1 \in \Omega_{y_1}(Y)$.

$(\hat{\alpha} * (f \circ \beta)) * \alpha$ and $g \circ \beta$ are isotopic in $\Omega_{y_1}(Y)$

1st attempt (F)

(4)
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$$\alpha * \underbrace{(h_s \circ \beta)}_{\begin{array}{l} \text{at } h_s(x) \\ \text{at } s \end{array}} * \alpha \quad [\text{undefined!}]$$

$\gamma_0 \sim \gamma_1$

Instead, let. $\gamma_s : [0,1] \rightarrow Y$

$$\gamma_s(t) = \alpha(s + (1-s)t). \quad s, t \in [0,1]$$

$$\gamma_s(0) = \alpha(s)$$

$$\gamma_0(t) = \alpha(t) \quad \gamma_1(t) = \alpha(1) = \gamma_1$$

Then $\overline{\gamma}_s * (h_s \circ \beta) * \gamma_s \in \Sigma_{Y_1}(Y)$. well-def
cont.

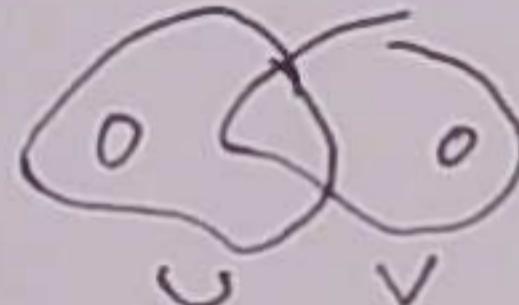
at $s=0 \quad \alpha * (f \circ \beta) * \alpha$

at $s=1 \quad e_{Y_1} * (g \circ \beta) * e_{Y_1} \simeq g \circ \beta$

□(law)

next Seifert - v. Kammer thm.

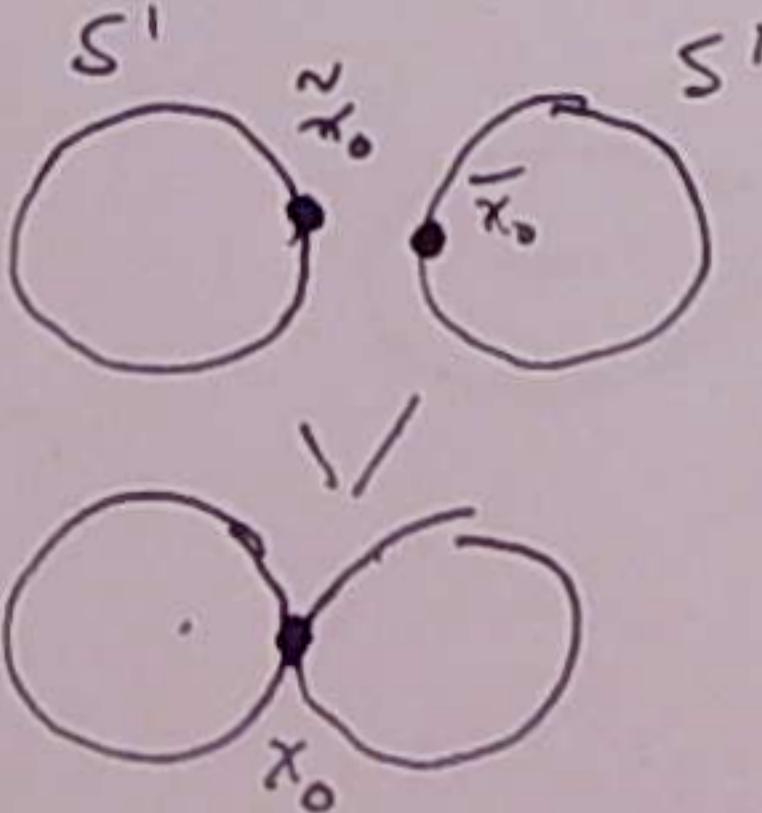
$$X = U \cup V$$



$U, V, U \cap V, X$ path-conn.

$\pi_1(U, x_0), \pi_1(V, x_0)$ generate $\pi_1(X, x_0)$ $x_0 \in U \cap V$.

basic case: $\pi_1(S^1 \vee S^1)$ is F_2 (free group on 2 gen.)



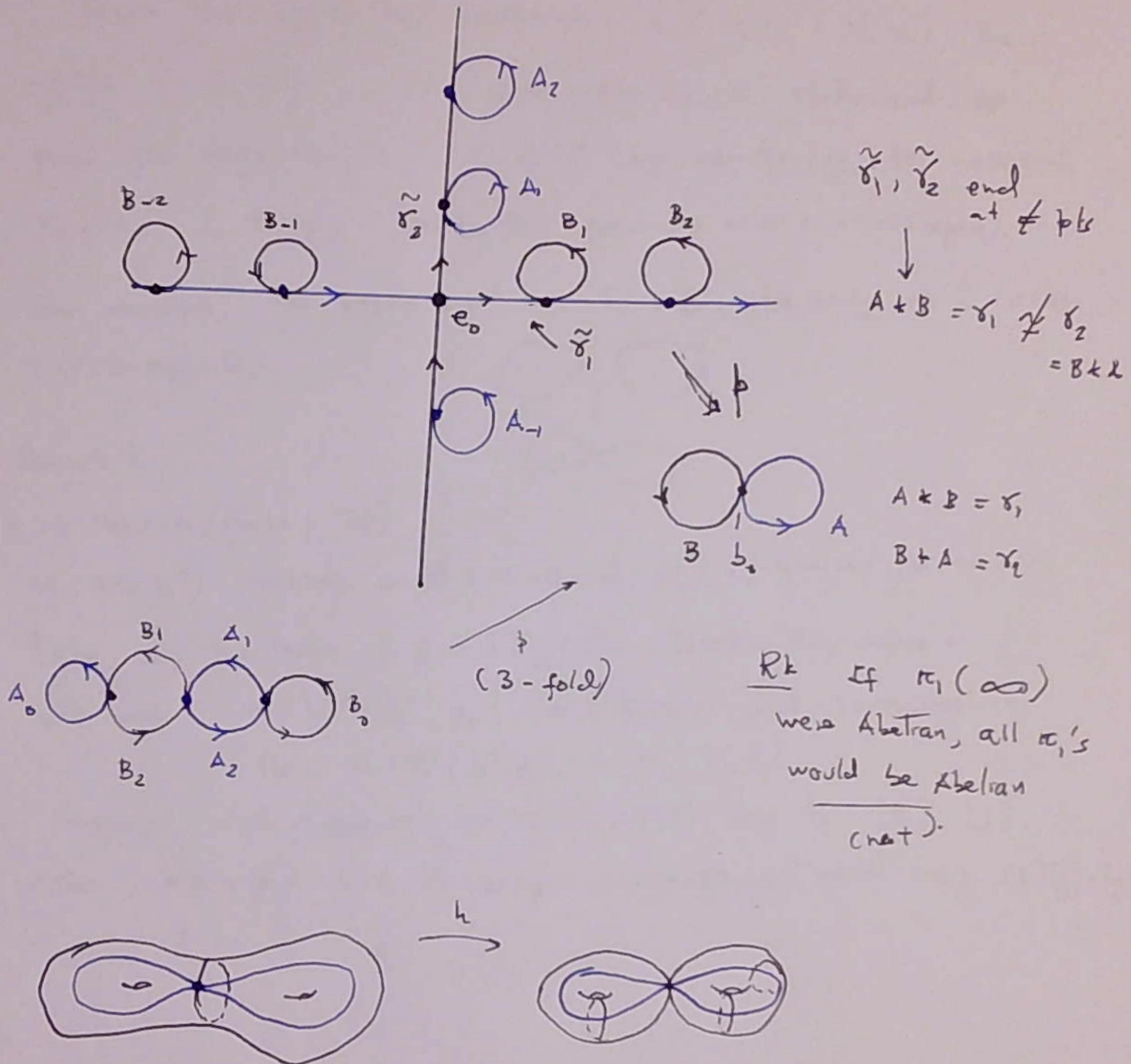
$(X, \tilde{x}_0), (Y, \tilde{y}_0)$

fig 1

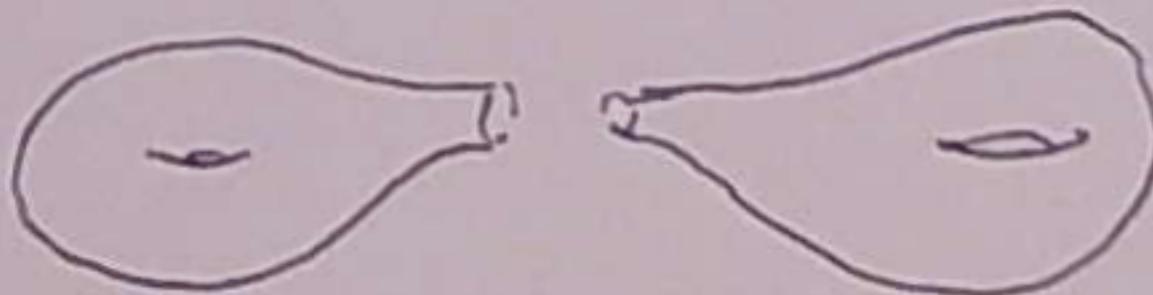
\downarrow
 $(X \vee Y, \tilde{x}_0)$

(5)

$\pi_1(S^1 \vee S^1)$ not abelian



$S^1 \vee S^1$ retract of $T^2 \# T^2$

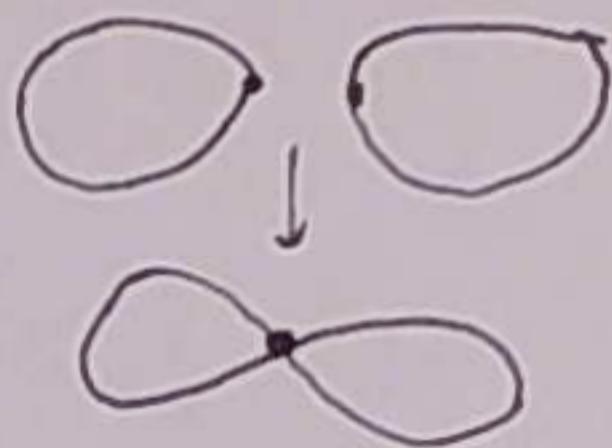


(6)

Remark 1 (cp. Fomenko/Fuchs p. 23)

Given two spaces w/ basepoints $(X, x_0), (Y, y_0)$, their wedge $X \vee Y$ is the space w/ basepoint obtained by from the disjoint union $X \sqcup Y$ by identifying the basepoints x_0, y_0 (Exercise: define this formally as a quotient space).

For example, the figure eight is $S^1 \vee S^1$, the wedge of 2 circles (w/ basepoints).



Remark 2

(cp. Fomenko/Fuchs p. 70)

$\pi_1(S^1 \vee S^1)$ abelian would have implied $\pi_1(X)$ abelian for all X :

take any two loops $f, g \in \Omega_{x_0}(X)$. Together they define a

cont. map $F : (S^1 \vee S^1, z_0) \rightarrow (X, x_0)$ and homomorphism

$$F_* : \pi_1(S^1 \vee S^1, z_0) \rightarrow \pi_1(X, x_0)$$

mapping α, β (generators of $\pi_1(S^1 \vee S^1)$) resp. to $[f], [g]$.

Then $\alpha\beta = \beta\alpha$ (and F_* being a homomorphism) would imply $[f][g] = [g][f]$