

## Borsuk-Ulam theorem

$$f: S^k \rightarrow \mathbb{R}^{k+1} \setminus \{0\} \text{ smooth}$$

Suppose  $f$  is odd  $f(-x) = -f(x)$ . Then  $\boxed{W_2(f, 0) = 1}$

Pf. (induction on  $k$ ). Assume true for  $k-1$  ( $k=1$ : later)

$$S^{k-1} \subset S^k \text{ (equator)}$$

$$\text{Let } g = f|_{S^{k-1}} \quad \hat{g}: S^{k-1} \rightarrow S^k \quad \hat{f}: S^k \rightarrow S^k$$

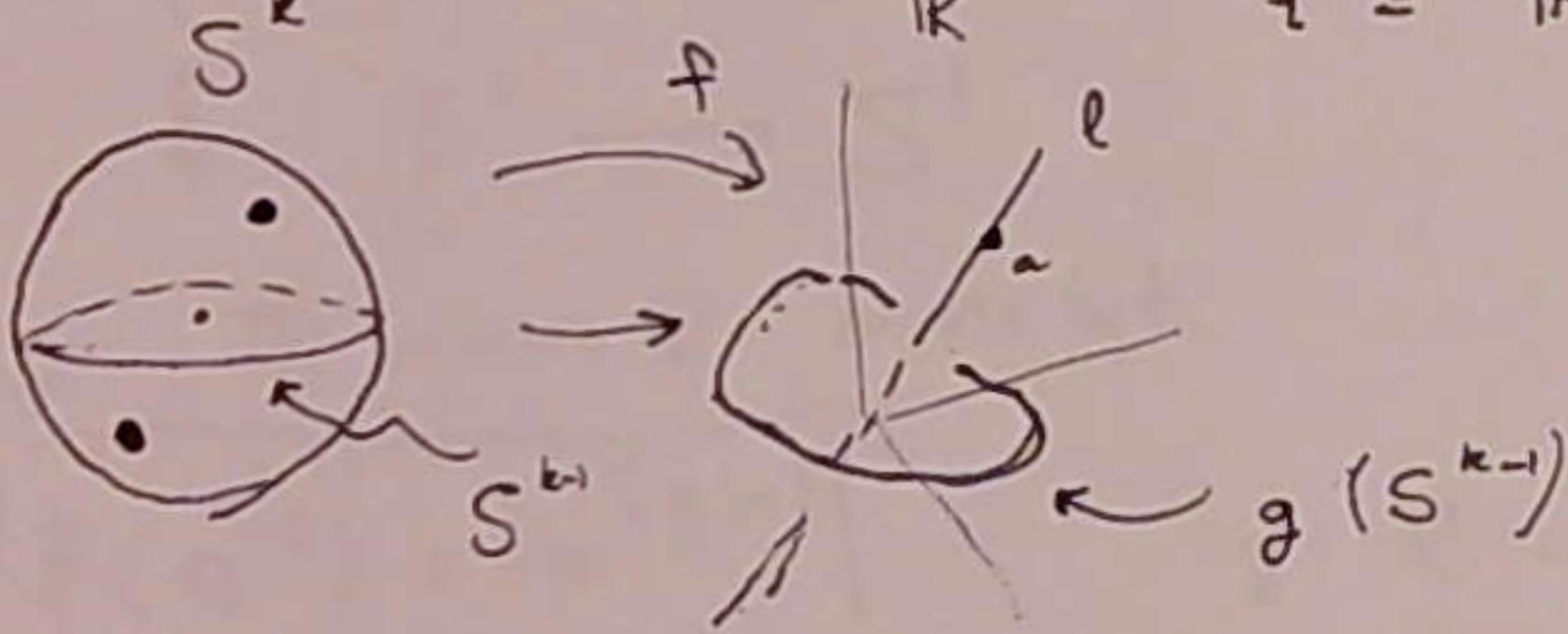
$$\hat{f}(x) = \frac{f(x)}{\|f(x)\|}$$

Let  $a \in S^k$  be a reg. value for  $\hat{f}, \hat{g}$ . ( $\hat{f}, \hat{g}$ : odd)

for  $\hat{g}$  this means:  $\pm a \notin \hat{g}(S^{k-1})$

$$\hat{g}(x) = \dots$$

$$l = \mathbb{R}a \notin g(S^{k-1})$$



$$f \neq l$$

$$df(x)[T_x S^k] + l = \mathbb{R}^{k+1}$$

$$\begin{aligned} W_2(f, 0) &\stackrel{\text{def}}{=} \deg_2(\hat{f}) = \# \hat{f}^{-1}(a) \pmod{2} \\ &= \frac{1}{2} \#(f^{-1}(l)) \quad \text{by symmetry.} \end{aligned}$$

$S^k_+$ : upper hemisphere

$$f_+: \text{restriction of } f \quad \# f_+^{-1}(l) = \frac{1}{2} \# [f(l)] \quad (\text{no need to count antipodal p.})$$

$$\text{so } \boxed{W_2(f, 0) = \# f_+^{-1}(l)}$$

Let  $V = l^\perp$  (orth. compl. in  $\mathbb{R}^{k+1}$ ) ( $\Rightarrow \dim V = k$ )

$$\pi: \mathbb{R}^{k+1} \rightarrow V \text{ orth. proj.}$$

conclude  $\pi \circ g: S^{k-1} \rightarrow V \quad 0 \notin \text{Im}(\pi \circ g)$  (case  $g(S^{k-1}) \cap l = \emptyset$ )

By ind. hyp:  $\boxed{W_2(\pi \circ g, 0) = 1}$

(2)

Since  $f_+$  and  $\ell$  on  $S_+^k$ ,  $\pi \circ f_+: S_+^k \rightarrow V$  is transv. to  $\{0\}$   
 $\downarrow$  ( $0 \in V$  is a reg. value)

note  
 $\pi \circ f_+$  extends  $\alpha \circ g$

from  $S^{k-1} = \partial S_+^k$  to  $S_{k+}^k$

thus (prev. th.).

$d(\pi \circ f_+)(x): T_x S^k \rightarrow V$  is onto  
 $(d\pi \circ f)(df)(x)$  (i.e. iso)

[follows from  $\oplus$ ]

$$W_2(\pi \circ g, 0) = \# \left[ (\pi \circ f_+)^{-1}(0) \right] = \# [f_+^{-1}(l)]$$

$$\stackrel{\text{so}}{=} W_2(f, 0) \# [f_+^{-1}(l)] = W_2(\pi \circ g, 0) = 1 \quad \hookrightarrow \text{ind. hyp.} \quad \square$$

Case  $k=1$   $\oplus$  see also p. 5

$f: S^1 \rightarrow \mathbb{R}^2$  smooth, "odd"  $f(-x) = -f(x)$

$\hat{f}: S^1 \rightarrow S^1$

$g: \mathbb{R} \rightarrow \mathbb{R}$  smooth lift

$$\boxed{\hat{f}(e^{it}) = e^{ig(t)}}$$

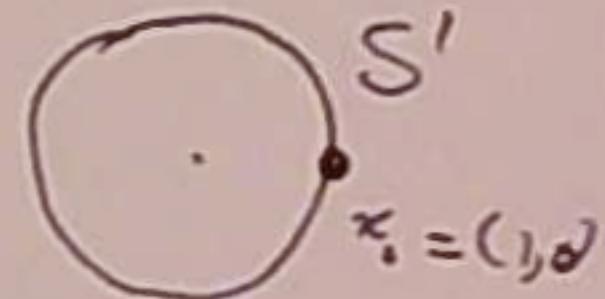
$$\hat{f} \circ g = f \circ \hat{f}$$

$\phi: \mathbb{R} \rightarrow S^1$  wavy

$$\phi(t) = e^{it}$$

$$\xrightarrow{\circ} \mathbb{R}$$

$$\begin{array}{ccc} g: \mathbb{R} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{\hat{f}} & S^1 \end{array}$$



$$g(0 + 2\pi) = g(0) + 2\pi q \quad (\text{for some } q \in \mathbb{Z}).$$

$$g(t + 2\pi) = g(t) + 2\pi q \quad t \in \mathbb{R} \quad (\text{from uniqueness of lifts})$$

this implies if  $y \in S^1$  is a reg. value of  $\hat{f}$ ,  $\#\hat{f}^{-1}(y) = q$

$$W_2(f, 0) = \deg_2 f = q \in \mathbb{Z} \quad \swarrow \quad f(-e^{it}) = -\hat{f}(e^{it})$$

If  $f$  is odd

$$g(t + \pi) = g(t) + \pi q \quad q \in \mathbb{Z} \quad \text{odd}$$

$$g(t + 2\pi) = g(t) + 2\pi q \quad q \text{ odd}$$

$$\text{so } W_2(f, 0) = q \text{ (odd)}, \text{ so } \equiv 1 \pmod{2}. \quad (\text{conclusion if } k=1)$$

Cor. Given  $k$  fns on  $S^k$   $g_i : S^k \rightarrow \mathbb{R}$   $i = 1, \dots, k$  (mostly)  
 $\exists p \in S^k$  s.t.  $g_i(-p) = g_i(p)$   $i = 1, \dots, k$  //

### Fundamental grp.

The  $X = U \cup V$   $U, V$  open in  $X$  (path-connected)

suppose  $U \cap V$  is connected (path-)

$$x_0 \in U \cap V$$

Then  $\pi_1(X, x_0)$  is generated by  $\pi_1(U, x_0)$  and  $\pi_1(V, x_0)$   
 $\tilde{G}$   $\parallel$   $H_U$   $\parallel$   $H_V$

this means any  $g \in G$  can be written as a finite product ("word")

$$g = h_1 * h_2 * \dots * h_n \quad \text{each } h_i \text{ in } H_U \text{ or } H_{U \cap V} \quad (\text{note } H_U \subset G, H_V \subset G)$$

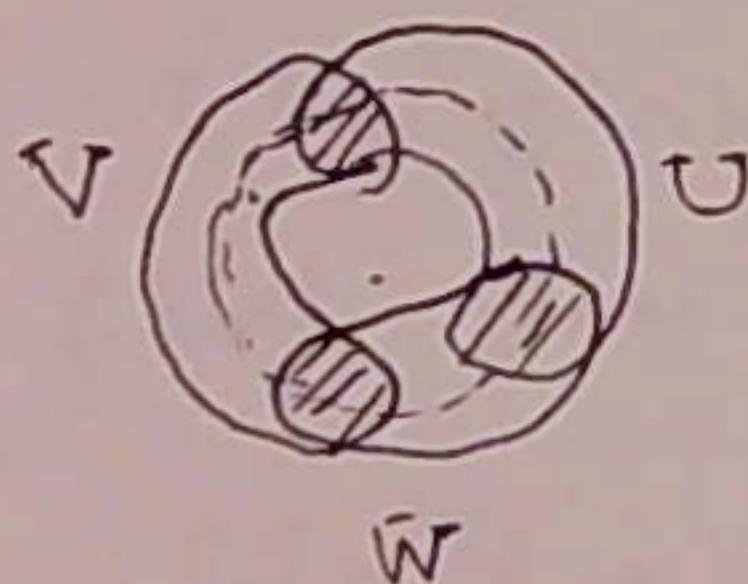
subgroup  
by inclusion

Rk Not true for 3 sets  $U, V, W$

(w/ pairwise intersections connected)

$$X = U \cup V \cup W$$

not simply-connected.



$$\begin{matrix} U & \rightarrow & X \\ V & \rightarrow & X \end{matrix}$$

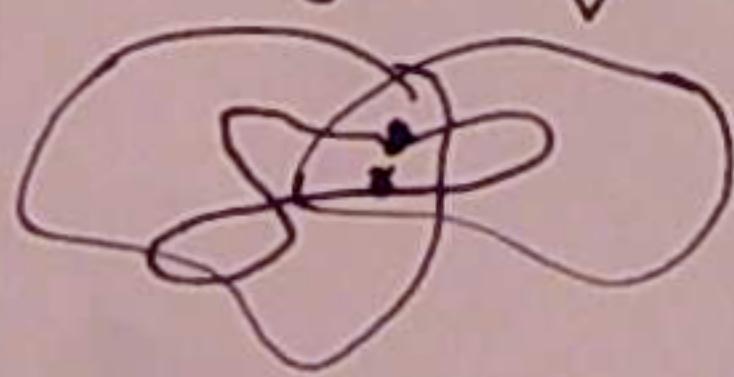
Proof int: any loop in  $X$  w/o point  $x_0$  is homotopic to one of the form:

$$g_1 * (g_2 * (g_3 * \dots * g_n)) \dots \quad \text{each } g_i : \text{loop (bp } \sim \text{)} \text{ in } U \text{ or in } V.$$

Let  $f$  be a loop at  $x_0$ .

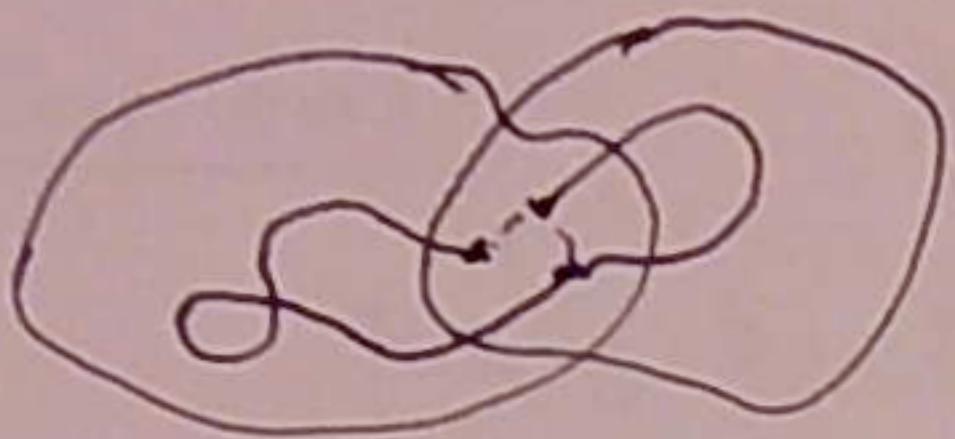
①  $\exists a_1 < \dots < a_n$  partition of  $[0, 1]$  s.t.  $f|_{[a_{i-1}, a_i]}$  contained in  $U$  or in  $V$

$$U \quad V \quad \text{and } f(a_i) \in U \cap V$$



$$b_1 < \dots < b_m \quad f|_{[b_{i-1}, b_i]} \subset U \text{ or } V \quad (\text{Lebesgue number})$$

If  $f|_{[b_{i-1}, b_i]}, f|_{[b_i, b_{i+1}]}$  are in the same set ( $U$  or  $V$ ) collapse the intervals.

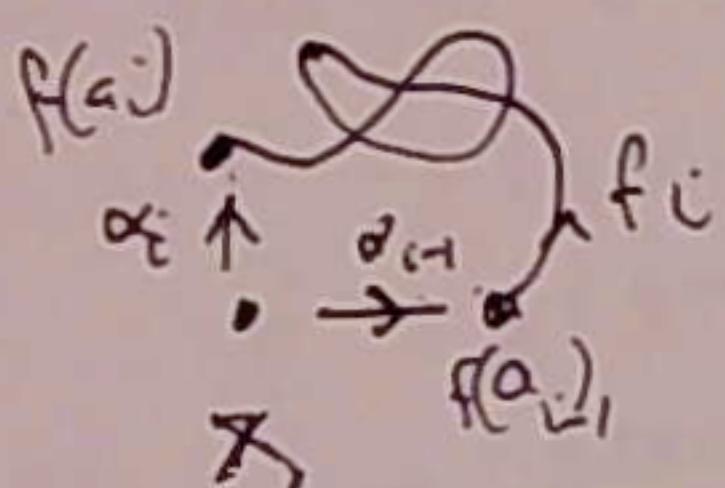


Now let  $f_i : [0,1] \rightarrow X$  reparametrize  $f|_{[a_{i-1}, a_i]}$

$$[f] = [f_1] + \cdots + [f_n] \quad (\text{path homoty}).$$

choose a path  $\alpha_i$  in  $U \cap V$  from  $x_0$  to  $f(a_i)$  ( $U \cap V$  conn)

Let  $g_i = (\alpha_{i-1} * f_i) * \bar{\alpha}_i$  (loop at  $x_0$ ).



$$[f] = [g_1] + \cdots + [g_n]$$

(in  $\pi_1(X, x_0)$ )

$g_i \in \pi_1(U, x_0)$  or  $\pi_1(V, x_0)$ .

Cor:  $S^n$  simply-conn (n ≥ 2).

$$U = S^n \setminus N \quad V = S^n \setminus S$$

(both homeo. to  $\mathbb{R}^n$  via stereogragy.)

$$U \cap V = S^n \setminus \{N, S\} \approx \mathbb{R}^n \setminus \{0\} \quad \text{connected} \quad (n \geq 2)$$

(homeo).

so the  $\cap$  implies  $S^n = U \cup V$  is simply-conn.

Case  $k=1$  :  $f : S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$  odd  $\Rightarrow w_2(f, 0) = 1 \pmod{2}$

1 Passing to  $\hat{f} : S^1 \rightarrow S^1$  (odd), we claim  $\deg_2 \hat{f} = 1$ .

2 Let  $x_0 = (1, 0) \in S^1$ ,  $\hat{f}(x_0) = y_0 = e^{it_0}$  for a unique  $t_0 \in [0, 2\pi]$

Let  $\alpha : [0, 2\pi] \rightarrow S^1$  be the closed curve in  $S^1$ :  $\alpha(t) \in \hat{f}(e^{it})$   
 $(\alpha$  is a loop at  $y_0$ )  $\alpha(0) = \alpha(2\pi)$ .

Consider the covering  $p : \mathbb{R} \rightarrow S^1$   $p(t) = e^{it}$ .

$\alpha$  lifts to a unique  $g : [0, 2\pi] \rightarrow \mathbb{R}$  s.t.  $p(g(0)) = \alpha(0) = y_0$ .

i.e.  $p(g(t)) = \alpha(t) \quad t \in [0, 2\pi]$

or  $e^{ig(t)} = \alpha(t)$

Then  $e^{ig(2\pi)} = \alpha(2\pi) = \alpha(0) = y_0$ , so  $g(2\pi) = g(0) + 2\pi q$ .

3 We show  $\hat{f}$  is homotopic to the map  $z \mapsto z^q$  ( $S^1 \rightarrow S^1$ ).

Let  $g_s(t) = (1-s)g(t) + stq \quad s \in [0, 1], t \in [0, 2\pi]$

$$\text{Then } g_s(2\pi) = (1-s)g(2\pi) + 2\pi s q$$

$$= (1-s)g(0) + 2\pi(1-s)q + 2\pi s q = (1-s)g(0) + 2\pi q$$

$$= g_s(0) + 2\pi q$$

so  $\hat{f}_s(z) = p(g_s(t)) \quad (t \in [0, 2\pi], s \in [0, 1])$  defines  $\hat{f}_s : S^1 \rightarrow S^1$

$$\hat{f}_0(e^{it}) = p(g(t)) = \alpha(t) = \hat{f}(e^{it}) \quad \text{so } \hat{f}_0 = \hat{f} \quad (\text{smooth})$$

$$\hat{f}_1(e^{it}) = e^{ig(0)} \quad \text{or} \quad \hat{f}_1(z) = z^q, \quad q \in S^1.$$

Thus  $\hat{f}$  and  $z \mapsto z^q$  are homotopic maps of  $S^1$ , and have the same  $\deg_2$ .

4 Now  $\deg_z(\omega_q) = q \pmod{2}$ , since  $\#^{-1}\omega_q(y) = q \quad \forall y \in S^1$ . Thus  $\deg_2(\hat{f}) = q \pmod{2}$

5 From  $e^{ig(t)} = \hat{f}(e^{it})$ , in part:  ~~$e^{ig}$~~  (now use  $\hat{f}$  odd)

$$e^{ig(t+\pi)} = \hat{f}(e^{i(t+\pi)}) = -\hat{f}(e^{it}) = -e^{ig(t)} = e^{i(g(t)+\pi)} \quad \text{so :}$$

$$g(t+\pi) = g(t) + \pi + 2\pi k \quad \text{for some } k \in \mathbb{Z} \quad (\text{indp of } t), \text{ extends}$$

$$\text{For } t=0 : g(\pi) = g(0) + \pi + 2\pi k \quad \text{For } t=\pi \quad g(2\pi) = g(\pi) + \pi + 2\pi k$$

$$\text{thus } g(0) + 2\pi q = g(2\pi) = g(0) + 2\pi + 4\pi k, \quad \text{or } q = 1 + 2k \quad (k \in \mathbb{Z})$$

$$\text{Thus } \deg_2(\hat{f}) = 1 \pmod{2}.$$

$$\text{so } q \equiv 1 \pmod{3}$$