

17) Fix  $z \in \mathbb{R}^n \setminus X$  ( $X \subset \mathbb{R}^n$  cpt. hypersurface)

For  $v \in S^{n-1}$ , consider the ray  $r_v = \{z + tv; t \geq 0\}$ .

Claim  $r_v \cap X \iff v$  is a regular value of  $u: X \rightarrow S^{n-1}$   
(direction map)

Pf Translating the origin to  $z$ , may assume  $z = 0, 0 \notin X$

$$r_v = \{tv; t \geq 0\}$$

(a)  $r_v \cap X$  means  $r_v \cap X \neq \emptyset$  or  $\forall x \in r_v \cap X \quad \langle v \rangle + T_x X = \mathbb{R}^n$   
(or i.e.  $\forall x \in r_v \cap X \quad v \notin T_x X$ )

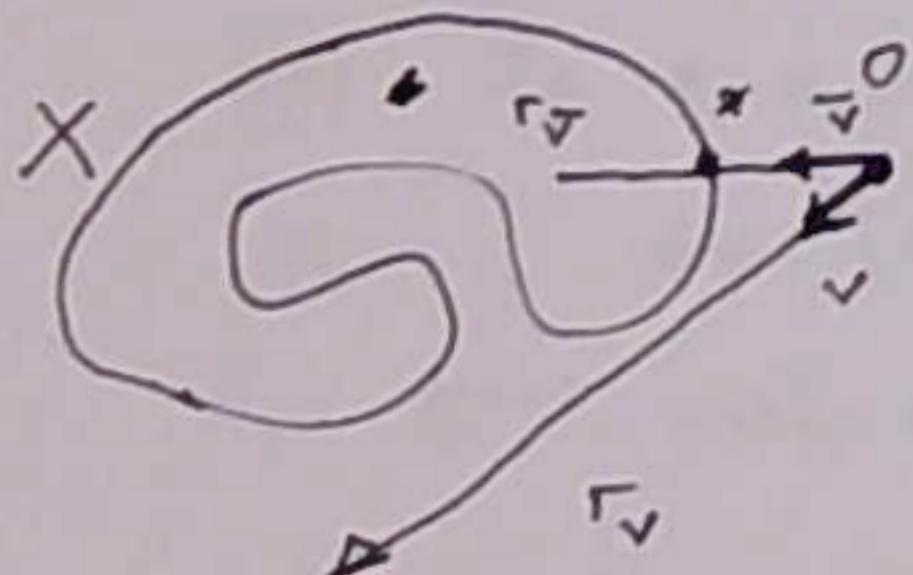
$$u(x) = \frac{x}{\|x\|}, \text{ so for } x \in X:$$

$$du(x) = \frac{1}{\|x\|} P_{u(x)}^\perp : T_x X \xrightarrow{\subset \mathbb{R}^n} T_{u(x)} S^{n-1} = \langle u(x) \rangle^\perp \subset \mathbb{R}^n.$$

( $P_{u(x)}^\perp$ : orth. proj. from  $T_x X$  to the orth. complement of  $u(x)$ )

thus  $du(x)$  is iso  $\iff \text{Ker } du(x) = \{0\}$   
 $\iff u(x) \notin T_x X$ .

(b)  $v$  is a regular value of  $u \iff v \notin u(X)$  or  $v = u(x)$  and  
 $du(x)$  is iso



clearly  $v \notin u(X) \iff r_v \cap X = \emptyset$ .

and  $v = du(x), \quad \left. \begin{array}{l} \text{du}(x) \text{ iso} \end{array} \right\} \iff v \notin T_x X.$

( $x \in r_v \cap X \iff u(x) = v$ )

Thus (a)  $\iff$  (b), as claimed.

④ Calculation  $u = \frac{1}{\|x\|} x \implies du(x)[v] = d\left(\frac{1}{\|x\|}\right)\omega[v] + \frac{1}{\|x\|} v$

$$d\left(\frac{1}{\|x\|}\right)[v] = -\frac{1}{\|x\|^3} \langle x, v \rangle x$$

Thus  $du(x)[v] = \frac{1}{\|x\|} \underbrace{\left(v - \frac{\langle x, v \rangle x}{\|x\|}\right)}_{\text{orth. proj. of } v \text{ onto } \langle \frac{x}{\|x\|} \rangle^\perp} = \frac{1}{\|x\|} P_{u(x)}^\perp [v]$   
orth. proj. of  $v$  onto  $\langle \frac{x}{\|x\|} \rangle^\perp = u(x)^\perp$ .

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$r_v$  emanating from  $z_0$ .  $v \in S^{n-1}$   $z_0 \notin X$

$z_1 \in r_v$  ( $z_1 \neq z_0$ ,  $z_1 \notin X$ )

suppose  $r_v$  intersects  $X$  transversally in a nonempty finite set w/  $\ell$  elements on the segment from  $z_0$  to  $z_1$ .

Then  $w_2(X, z_0) = w_2(X, z_1) + \ell \pmod{2}$

Pf

$$w_2(X, z_1) = \#\mu_{z_1}^{-1}(v) \pmod{2} \quad (\text{since } v \text{ is a reg. value of } \mu_{z_1}: X \rightarrow S^{n-1})$$

$$\begin{aligned} \mu_{z_1}^{-1}(v) &= \{x \in X \mid \mu_{z_1}(x) = v\} \\ &= r_v(z_1) \cap X \\ &= (r_v(z_1) \cap X) \sqcup \{x_1, \dots, x_\ell\} \quad (\text{disjoint union}) \end{aligned}$$

where  $r_v(z_1)$  is the part of the ray starting at  $z_1$ ,

$$\begin{aligned} \text{and } \{x_1, \dots, x_\ell\} &= \underbrace{r_v[z_0, z_1]}_{\text{segment of } r_v \text{ from } z_0 \text{ to } z_1} \cap X \\ &\dots = \mu_{z_1}^{-1}(v) \sqcup \{x_1, \dots, x_\ell\} \end{aligned}$$

Thus  $w_2(X, z_0) = \#\mu_{z_0}^{-1}(v) = \#\mu_{z_1}^{-1}(v) + \ell = w_2(X, z_1) + \ell \pmod{2}$ .

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If  $z$  is large,  $w_2(X, z) = 0$ . It's enough to show  $\text{diam}[\mu(x)] \rightarrow 0$

For if  $\text{diam}[\mu(x)]$  is small enough, there will be open sets  $U$  as  $z \rightarrow \infty$  in  $S^{n-1}$  with empty preimage:  $\mu^{-1}(U) = \emptyset$ , so  $w_2(X, z) = 0$ .

To see this, let  $a, b \in X$ ,  $\mu(a) = \hat{a}$ ,  $\mu(b) = \hat{b}$

$$\hat{a} = \frac{z-a}{\|z-a\|}, \quad \hat{b} = \frac{z-b}{\|z-b\|}. \quad \text{Assume } \|z\| > 2M, \quad M = \max \{\|x\|, x \in X\}$$

$$\hat{a}-\hat{b} = \frac{1}{\|z-a\|\|z-b\|} \left( (z-a)\underbrace{\|z\|}_{\|z\|} \underbrace{\|z-b\|}_{1+\varepsilon(z,b)} - (z-b)\underbrace{\|z\|}_{\|z\|} \underbrace{\|z-a\|}_{1+\varepsilon(z,a)} \right)$$

$$\left| \frac{\|z-a\|}{\|z\|} \right| \leq 1 + \frac{\|a\|}{\|z\|} \quad \text{so} \quad |\varepsilon(z,a)| = \left| \frac{\|z-a\|}{\|z\|} - 1 \right| \leq \frac{\|a\|}{\|z\|} < \frac{1}{2} \quad \text{if } \|z\| > 2M$$

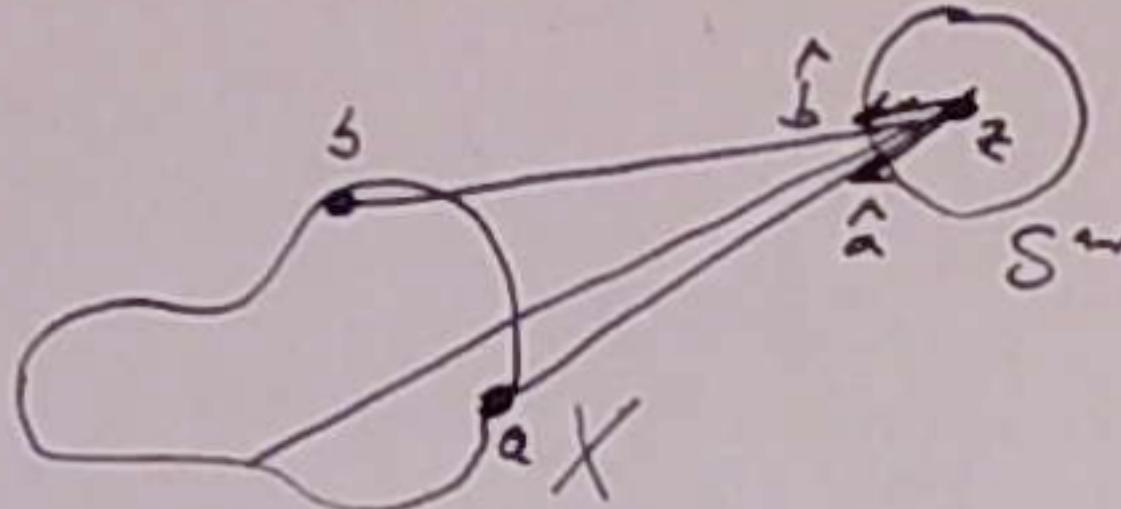
$$\text{while } \|z-a\|\|z-b\| \geq (\|z\|-\|a\|)(\|z\|-\|b\|) \geq M^2$$

$$\text{Thus } \hat{a}-\hat{b} = \frac{1}{\|z-a\|\|z-b\|} \left[ (b-a)\|z\| + (z-a)\|z\| \varepsilon(z,b) - (z-b)\|z\| \varepsilon(z,a) \right]$$

Hence:

$$\begin{aligned}\|\hat{a} - \hat{b}\| &\leq \|b-a\| \frac{\|z\|}{(\|z\|-M)^2} + \frac{\|z\|}{\|z-b\|} \cdot \frac{\|b\|}{\|z\|} + \frac{\|z\|}{\|z-a\|} \frac{\|a\|}{\|z\|} \\ &\leq \text{diam}(X) \frac{\|z\|}{(\|z\|-M)^2} + \frac{\|b\|}{\|z\|-M} + \frac{\|a\|}{\|z\|-M} \xrightarrow[\text{as } \|z\| \rightarrow \infty]{} 0\end{aligned}$$

clear geometrically:



summary

Jordan-Brouwer sep'n theorem

$X \subset \mathbb{R}^n$  cpt. hypersurface

$\Rightarrow \mathbb{R}^n \setminus X$  has two connected comps:  $D_0$  (unbounded) and  $D_1$ , with  $\partial D_1 = X$  (bounded)

outline of proof of JB

①  $\forall x \in X \quad \exists U_x \subset \mathbb{R}^n$  open  $\exists y \in U_x$  s.t.  $y$  can be connected to  $z$  by a path in  $\mathbb{R}^n \setminus X$ .

Thus  $\mathbb{R}^n \setminus X$  has at most 2 conn. components.

② If  $z_0, z_1$  are in the same c.c. of  $\mathbb{R}^n \setminus X$ ,  $W_2(X, z_0) = W_2(X, z_1)$ .

③ Characterization of  $W_2(X, z)$ : count (mod 2) the intersections of  $X$  w/ a ray from  $z$  intersecting  $X$  transversally.

④  $\mathbb{R}^n \setminus X = D_0 \sqcup D_1$ , where  $D_0 = \{z; W_2(X, z) = 0\}$   $D_1 = \{z; W_2(X, z) \neq 0\}$

From ③, (and prob. 10) both are non-empty. From ① both are open, hence both must be connected (otherwise: more than 2 c.c.).

$D_1$  bounded,  $D_0$  unbounded follows from prob. ⑩.

prob. 10. Take  $z_0$  far from  $X$  (so  $W_2(X, z_0) = 0$ ) and draw a ray from  $z_0$  intersecting  $X$  transversally in a non-empty set  $\{x_1, \dots, x_l\}$  and ending at  $z_1$ . Then,  $W_2(X, z_1) = l \pmod 2$ , so  $z_1 \in D_1$  (the bounded component) iff  $l$  is odd

prob 8