

Intersection mod 2

$f: X \rightarrow Y$ diff'ble X cpt, $\partial Y = \emptyset$ $Z \subset Y$ submanifold
 $\dim X + \dim Z = \dim Y$. $\partial Z = \emptyset$

If $f \neq Z$, $f^{-1}(Z)$: finite set.

$$I_2(f, Z) = \# f^{-1}(Z) \bmod 2$$

If $g: X \rightarrow Y$ diff'ble.

$\exists f: X \rightarrow Y$ homotopic to g and transv. to Z

(f_0, f_1 both transv. to Z ad $f_0 \sim f_1 \Rightarrow \# f_0^{-1}(Z) \equiv f_1^{-1}(Z) \bmod 2$)

$I_2(g, Z) \stackrel{\text{def}}{=} I_2(f, Z)$ for any $f \neq Z$, htopic to g

Prop. $g, f: X \rightarrow Y$ homotopic $\Rightarrow I_2(f, Z) = I_2(g, Z)$.

Boundary theorem

(recall If $X = \partial W$

Thm 1 $f: X \rightarrow Y$ diff'ble
 If f extends from
 X to W , then

$$\pm_2(f; Z) = 0$$

$f: X \rightarrow Y$ transv. to $Z \subset Y$
 If f extends to $g: W \rightarrow Y$ diff'ble
 then f admits an extension to W

(we still assume $\dim X + \dim Z = \dim Y$)

Pf. Suppose f admits a smooth extension to

$$F: W \rightarrow Y \quad (\text{may not } \# \neq 0). \quad \partial F = F|_{\partial W} = f.$$

Then $G: W \rightarrow Y$, $G \sim F$ $g = \partial G \neq Z$, $G \neq Z$.

Then $f \sim g$, so $\pm_2(g; Z) = I_2(f; Z) = \# f^{-1}(Z) \bmod 2$

$G^{-1}(Z)$ is a compact 1-diml submanifold w/bdy of X $\begin{bmatrix} \text{codim}_W G^{-1}(Z) \\ = \text{codim}_Y Z = \dim \\ = \dim W - 1 \end{bmatrix}$

$$\#\partial[G^{-1}(Z)] \equiv 0 \bmod 2 \quad \# g^{-1}(Z) \equiv 0 \bmod 2 \Rightarrow I_2(g; Z) = 0$$

$$\Rightarrow g^{-1}(Z) \equiv 0 \bmod 2$$

Mod 2 degree

X compact.

Thm $f: X \rightarrow Y$ diffble, $\underline{Y \text{ connected}}$, $\dim X = \dim Y$.

\nexists Let $y \in Y$.

Then $I_2(f, \{y\})$ is the same for all $y \in Y$. $\boxed{\begin{array}{l} \text{"mod 2} \\ \text{degree of } f \end{array}}$

Pf. ~~Let~~. If needed replace f by a homotopic smooth map \tilde{f} transversal to $\{y\}$ (i.e. y is a reg. value).

$$\tilde{f}^{-1}\{y\} = \{x_1, x_2, \dots, x_n\}$$

(for each x_i ~~if~~, $d\tilde{f}(x_i)$ is an isom. so \exists nbds U_i of x_i , V_i of y s.t. $\tilde{f}: U_i \rightarrow V_i$ is a diffed.).

so if $V = V_1 \cap \dots \cap V_n$ (nbld of y). ~~If~~ $z \in V$ $\# \tilde{f}^{-1}(z) = n$)

$\Rightarrow I_2(f, \{y\}) = I_2(\tilde{f}, \{y\}) = n$ (loc. constant).
constant \forall of y .

since Y is connected, $I_2(f, \{y\})$ is const in Y .

Cor. $f \sim g \Rightarrow \deg_2 f = \deg_2 g$. (since it is an intersection number)

Cor. $X = \partial W$ and $f: X \rightarrow Y$ ($\dim X < \dim Y$)

$\Rightarrow \deg_2 f = 0$. (since this is true for intersection number).

Ex. $z \xrightarrow{f} z^n$ in S^1

$$\# f^{-1}(w) = n \quad \forall w \in S^1 \rightarrow \deg_2 f = n \bmod(2)$$

n odd $\rightarrow f$ not homotopic to a const

$\rightarrow f$ does not extend smoothly to D .

(3)

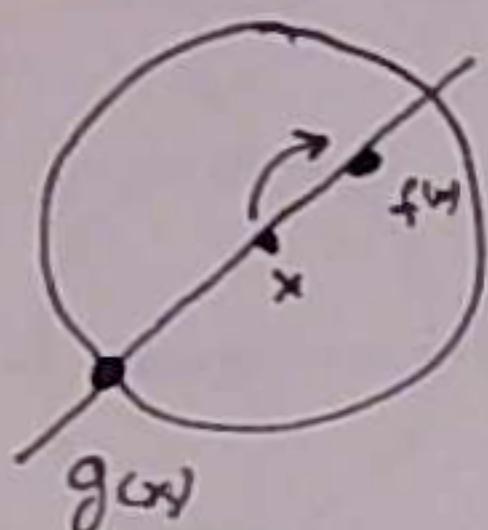
Recall

Brouwer fixed pt then $f: \mathbb{D}^n \rightarrow \mathbb{D}^n$ (smooth) has a fixed pt.

Lemma $\exists f: \mathbb{D}^n \xrightarrow[\text{(smooth)}]{\sim} S^{n-1}$ which is the identity on $\partial \mathbb{D} = S^{n-1}$

(Sard's thm + classif'n of 1-dm mfds)

Pf. of diff'ble Brouwer



If f has no fixed pts,

find $x \mapsto g(x)$ from \mathbb{D}^n to S^{n-1} ,
 $= id_{S^{n-1}}$ on S^{n-1} (contradicts lemma.)

Extension to cont. maps Let $f: \mathbb{D}^n \rightarrow \mathbb{D}^n$ be cont. Claim f has a fixed pt.
 By Stone - Weierstrass $\forall \varepsilon > 0$

$\exists p_1: \mathbb{R}^n \rightarrow \mathbb{R}^n$ polynomial map

s.t. $\|f - p_1\|_{\mathbb{D}^n} < \varepsilon$ (sup norm).

Then $p = \frac{1}{1+\varepsilon} p_1$ maps to \mathbb{D} : $p: \mathbb{D}^n \rightarrow \mathbb{D}^n$ smooth.

$$\forall x \in \mathbb{D} \quad \|p(x)\| = \frac{1}{1+\varepsilon} \|p_1(x)\| \leq \frac{1}{1+\varepsilon} (\underbrace{\|f(x)\|}_{\leq 1} + \varepsilon) \leq 1$$

p has a fixed pt. $x_0 \in \mathbb{D} : p(x_0) = x_0$.

Suppose f doesn't. Then $\inf_{x \in \mathbb{D}^n} \|f(x) - x\| = \mu > 0$
 Choose $\varepsilon < \mu$

$$0 = \|p(x_0) - x_0\| \geq \|f(x_0) - x_0\| - \|p(x_0) - f(x_0)\| \geq \mu - \varepsilon > 0$$

Contradiction

Rk The property " f does not have a fixed pt" is C^0 open in the uniform topology
 (on $c\bar{c}$ compact mfds).

(4)

Fund. Thm. of Algebra

$p: \mathbb{C} \rightarrow \mathbb{C}$ non-const poly

$\Rightarrow p$ has a root.

$$P: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$$

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

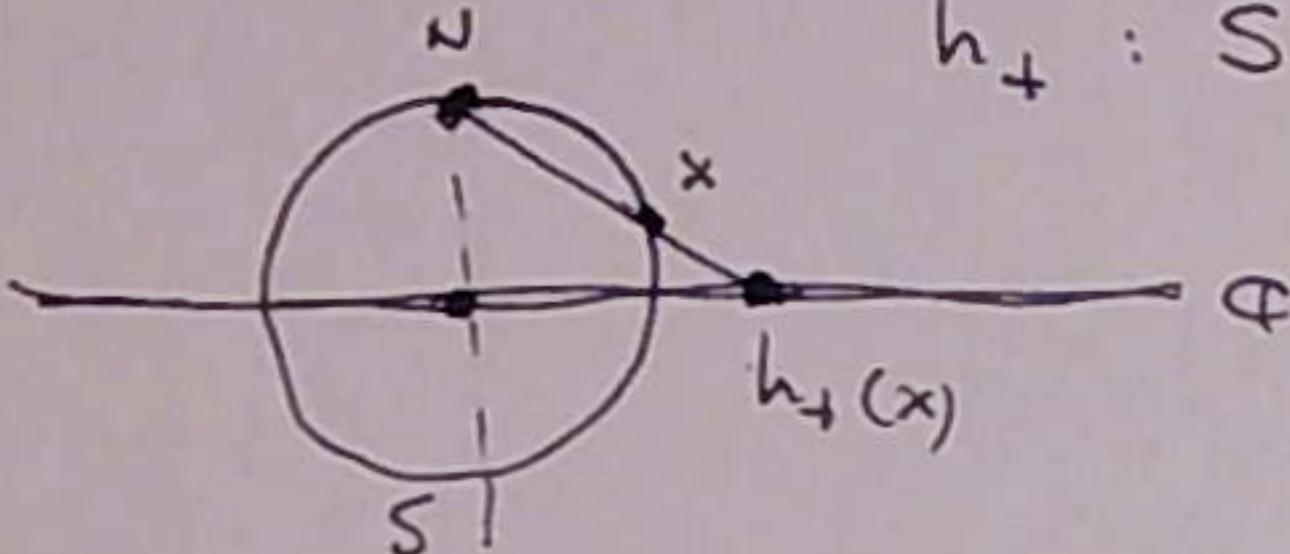
$$P(\infty) = \infty$$

(one-pt or Alexander compactification).

(cont extension of p to $\hat{\mathbb{C}}$).

$\hat{\mathbb{C}}$ homeo to S^2

$$h_+: S^2 \hookrightarrow \mathbb{R}^3$$



$$h_+: S^2 \hookrightarrow \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \quad (N \mapsto \infty)$$

stereographic proj... : homeomorphism

$$f: S^2 \rightarrow S^2 \xrightarrow{\text{smooth nat.}} \text{(check in local charts).}$$

crit pt

$$f = h_+^{-1} \circ P \circ h_+$$

$$h_+: S^2 - \{N\} \rightarrow \mathbb{R}^2$$

$C(f)$ is finite

$$h_-: S^2 - \{S\} \rightarrow \mathbb{R}^2$$

(zeros of $f'(z)$, and ∞)

$$h_+ h_-^{-1}(z) = \frac{z}{|z|^2}$$

hence the set of reg ~~pt~~ values of f on S^2

is $S^2 - \{w_1, \dots, w_N\}$ (connected).

~~Thus~~ the # $f^{-1}(y)$ is locally constant in the set of reg. vals y (seen earlier). thus constant over the set of reg. vals.

This number cannot be zero (since $f(S^2)$ is not finite)

so f is onto (note $f(S^2)$ is compact) //

i.e. P has a root.