# Sard's theorem

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ of class $C^r$.

$C = \{ x \in \mathbb{R}^n \mid \operatorname{rk}(df(x)) < p \}$ ← crit. pts of $f$ $\quad (n < p \Rightarrow C = \mathbb{R}^n)$

**Thm** $f(C)$ is a set of measure 0 in $\mathbb{R}^p$

($\uparrow$ set of crit. values)

**Def** $A \subset \mathbb{R}^p$ has measure 0 if $\forall \varepsilon > 0$ $\exists$ $A \subset \bigcup_{i=1}^{\infty} Q_i$ $Q_i \subset \mathbb{R}^p$ cubes

s.t. $\sum_{i=1}^{\infty} \operatorname{vol}_p(Q_i) < \varepsilon$.

**Ex.** (i) A countable $\longrightarrow$ meas$(A) = 0$ ↘ (cover each $x_i \in A$ by a cube w/ edge $\frac{\varepsilon}{2^i}$ $\quad \sum_{i=1}^{\infty} \frac{\varepsilon^p}{2^{ip}} < \varepsilon$)

(ii) countable unions of sets of measure 0 have measure 0

[ cover $A_i$ w/ total vol. $\frac{\varepsilon}{2^i}$ (by cubes) $\longrightarrow$ cover $\bigcup_{i=1}^{\infty} A_i$ w/ total volume $\sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon$. ]

(iii) $S \subset \mathbb{R}^p$ cube.

$S \subset \bigcup_{i=1}^{\infty} Q_i$ $Q_i$ cube. $\Rightarrow \operatorname{vol}(S) \leq \sum_i \operatorname{vol}(Q_i)$ (see [G-P])

**thus:** A set of measure 0 has empty interior.

A property P holds "a.e." (almost everywhere) if the set where it doesn't hold has measure 0

( P holds a.e. $\Rightarrow$ P holds on a dense set)

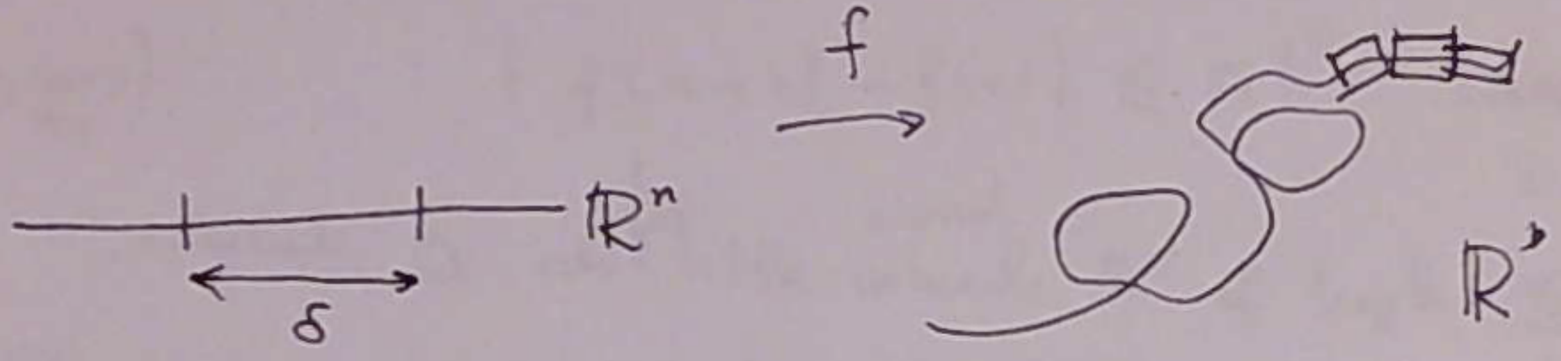
(TE) Sard's thm $\Rightarrow$ set of regular values is dense in $\mathbb{R}^p$.

(iv) ✱ $f: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ loc. Lipschitz (e.g. $C^1$)

$A \subset \mathbb{R}^n$ has meas. 0 $\Rightarrow$ $f(A) \subset \mathbb{R}^p$ has meas. 0.

Proof of Sard's thm- We look at each essentially different special case

**Case 1** $f: \mathbb{R}^n \xrightarrow{C^1} \mathbb{R}^p$ $n < p$ want $f(\mathbb{R}^n)$ has meas. 0 in $\mathbb{R}^p$




$Q^n$ : cube in $\mathbb{R}^n$ w/ edge $\delta$ goal cover $f(Q^n)$ by cubes in $\mathbb{R}^p$ w/ small total volume.

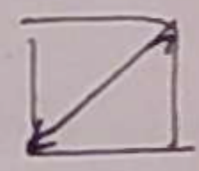
$x \in Q^n$

and $x+h \in Q^n \Rightarrow \|f(x+h) - f(x)\| \leq C\|h\|$ (*) ($C$ dep. on $Q^n$ and on $f$

($C^1$ maps are loc. Lipschitz)

Fix $L >> 1$

subdivide $Q^n$ into $L^n$ little cubes of edge $\frac{\delta}{L}$.



$Q'$

$x \in Q'$
$x+h \in Q'$ (*) $\Rightarrow$ diam $f(Q') < C\sqrt{p}\,\frac{\delta}{L}$

so $f(Q')$ is contained in a cube in $\mathbb{R}^p$ w/ edge $2\sqrt{p}C\frac{\delta}{L}$,

so $f(Q)$ is contained in the union of the $f(Q')$, i.e. in a union of cubes in $\mathbb{R}^p$ w/ total volume $M \cdot \frac{L^n}{L^p}$

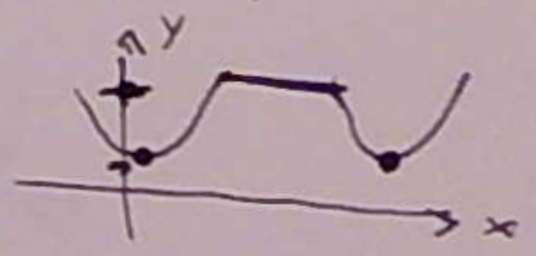
$$\text{vol} = \frac{(2\sqrt{p}C\delta)^p}{L^p} = \frac{M}{L^p}$$

Since $n < p$, this can be made $< \varepsilon$ taking $L$ suff. large.

Thus $f(Q^n)$ has measure 0 Cover $\mathbb{R}^n$ by countably many $Q$.

to see $f(\mathbb{R}^n)$ has measure 0 (using (ii) from p-①).

**Case #2** $f: \mathbb{R} \longrightarrow \mathbb{R}$ (say smooth)




Rk $C$ can be any closed set subset of $\mathbb{R}$

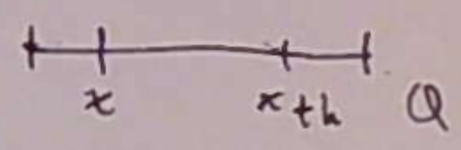
$C = \{x \in \mathbb{R} \mid f'(x) = 0\}$ Claim $f(C)$ has measure 0 ($f$ at least $C^1$)

critical set

**Case 2** $f: \mathbb{R} \to \mathbb{R}$ (cont'd)

$Q \subset \mathbb{R}$ closed interval of length $\delta$.

Let $x \in C \cap Q$ (crit. point in $Q$).




Rk $\exists$ a $c > 0$ s.t. if $x+h \in Q$

(Taylor's thm) $|f(x+h) - f(x)| \leq c h^2$ since $f'(x) = 0$ $(f \in C^1)$

c dep. on $Q$

subdivide $Q$ into $L$ little closed intervals $Q'$ of length $\frac{\delta}{L}$ $(L >> 1)$

$x \in Q'$, $x+h \in Q'$: $|f(x+h) - f(x)| \leq \frac{c\delta^2}{L^2}$

$\operatorname{diam} f(Q') \leq \frac{c\delta^2}{L^2}$, so $f(Q')$ is contained in a closed interval of length $\frac{c\delta^2}{L^2}$

$f(Q)$

Thus $f(C)$ is contained in a union of closed intervals w/ total length $L \cdot \frac{c\delta^2}{L^2} < \varepsilon$ for $L$ suff. large.

$f(C)$ has measure 0

thus $f(C)$ has measure 0

The critical set $C$ can be covered by ctbly many (interval) cubes $Q$

so $f(C)$ has meas. 0 in $\mathbb{R}$ (since each $f(C \cap Q)$ has meas. 0)

Remark The same argument shows for $f: \mathbb{R}^n \xrightarrow{C^1} \mathbb{R}^n$

letting $C_1 = \{x \in \mathbb{R}^n \mid df(x) = 0\}$ we have $f(C_1)$ has meas. 0 in $\mathbb{R}^n$.

but this is not the same as $C = \{x \in \mathbb{R}^n \mid \operatorname{rank} df(x) < n\}$ $(C_1 \in C)$.

**Case 3** $f: \mathbb{R}^2 \to \mathbb{R}^2$ know $f(C_1)$ has meas. 0 in $\mathbb{R}^2$

need $f(C \setminus C_1)$ has measure 0 in $\mathbb{R}^2$.

say $f(x) = (f_1(x), f_2(x))$.

Let $\bar{x} \in C \setminus C_1$; say $\frac{\partial f_1}{\partial x_1}(\bar{x}) \neq 0$

Consider the map $h(x) : (f_1(x), x_2)$ $h: \mathbb{R}^2 \to \mathbb{R}^2$

Then $dh(\bar{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}) & \frac{\partial f_1}{\partial x_2}(\bar{x}) \\ 0 & 1 \end{bmatrix}$ $dh(\bar{x}) \in L(\mathbb{R}^2)$ is 1-1

by the IFT $\exists$ $V$ (nbd of $\bar{x}$ in $\mathbb{R}^2$) , $V' \subset \mathbb{R}^2$ open

s.t. $h: V \xrightarrow{\sim} V'$ is a diffeo.

Consider $g = f \circ h^{-1} : V' \longrightarrow \mathbb{R}^2$. $h^{-1}: V' \to V$

crit pts of $g$ $\to$ $C_g = h(C_f \cap V)$

$g = (g_1(y), g_2(y))$ $y \in V'$

so $g(C_g) = f(C_f \cap V)$ (*)

crit. values of $g$ want. $g(C_g)$ has meas. 0 in $\mathbb{R}^2$.

Claim $g_{t,1}(t, y_2) = t$

「Pf. $g_1(t, y_2) = f_1(h^{-1}(t, y_2))$

$(t, y_2) \in V'$

$(t, y_2) = h(x) = (f_1(x), x_2) \longrightarrow \begin{cases} t = f_1(x) \\ x_2 = y_2 \end{cases}$

$h^{-1}(t, x_2) = x = (x_1, x_2)$

$t \in \mathbb{R}$ $f_1(h^{-1}(t, x_2)) = f_1(x) = t$ i.e. $g_1(t, y_2) = t$.」

$g_t$ maps $(\{t\} \times \mathbb{R}) \cap V'$ to $\{t\} \times \mathbb{R}$ ($g_t$ is a one-variable map)

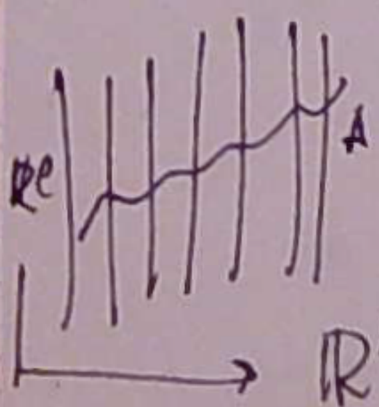
$g_t : U \subset \mathbb{R} \longrightarrow \mathbb{R}$

the set of crit values of $g_t$ = set of crit values of $g|_{(t \times \mathbb{R}) \times V'}$ has meas 0 (by case (ii)).

Fubini's thm (works for maps $\mathbb{R}^n \to \mathbb{R}^p$, $n \le p$, by induction on $n$)

$\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$ $A \subset \mathbb{R}^n$ closed

$A \cap (t \times \mathbb{R}^l)$ has meas 0 for all $t \in \mathbb{R}^k$ $\Rightarrow$ $A$ has meas 0 in $\mathbb{R}^n$




so $g(C_g)$ (crit values of $g$) has meas 0 in $\mathbb{R}^2$

(A) to $f(C_f \cap V)$ has meas. 0.

cover $C_f$ & $C_1$ by ctbly many $V$ to see $f(C_f \setminus C_1)$ has

Rk. This argument works for maps $\mathbb{R}^n \to \mathbb{R}^p$, by induction on $n$. (i.e. we get $f(C_f \setminus C_1)$ has measure 0, if for all $g: \mathbb{R}^{n-1} \to \mathbb{R}^p$, $g(C_g)$ has measure 0).

**Case 4** $f: \mathbb{R}^n \to \mathbb{R}$

$$C_f = \{x \in \mathbb{R}^n \mid df(x) = 0\} = C_1$$

Let $Q^n$ be a cube of edge $\delta$, $x \in C_f \cap Q^n$.

$|f(x+h) - f(x)| \leq C\|h\|^2$ $\quad x, x+h \in Q^n$ ($C$ dep. on $f, Q$).

subdivide $Q$ into $L^n$ cubes $Q'$ of edge $\frac{\delta}{L}$ ($L >> 1$)

$f(Q')$ has diam $\leq C\frac{\delta^2}{L^2}$, hence contained in an interval of edge $C\frac{\delta^2}{L^2}$.

**Problem** total volume of $f(Q) \leq C\frac{\delta^2}{L^2} \cdot L^n$ which does not $\to 0$ as $L \to \infty$.

say $n = 2$ Consider $C_2 = \{x \in \mathbb{R}^2;\ df(x) = 0,\ d^2f(x) = 0\}$

Let $x \in C_2 \cap Q^n$ (edge $(Q) = \delta$) w/ $x + h \in Q^n$ $\quad |f(x+h) - f(x)| \leq C\|h\|^3$ (need $f \in C^3$, $C$ dep. on $Q, f$)

Take $L^2$ small cubes $Q'$ w/ edge $\frac{\delta}{L}$

$f(Q')$ contained in an interval w/ edge $C\frac{\delta^3}{L^3}$.

$\Rightarrow$ $f(Q)$ covered by intervals w/ total length $(C\delta^3)\frac{L^2}{L^3} \to 0$.

so $f(C_2 \cap Q^n)$ has meas. 0, so $f(C_2)$ has meas. 0.

need $f(C_1 \setminus C_2)$ has meas. 0.

Let $\bar{x} \in C_1 \setminus C_2$ so $\frac{\partial f}{\partial x^1}(\bar{x}) = 0$, $\frac{\partial f}{\partial x^2}(\bar{x}) = 0$ but (say) $\frac{\partial^2 f}{\partial x_1^2}(\bar{x}) \neq 0$

Let $w(x) = \frac{\partial f}{\partial x^1}(x)$, define

$h: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad h(x) = (w(x), x_2^2)$ $\quad dh(\bar{x}) \in L(\mathbb{R}^2)$ ISOM.

so $h: V \xrightarrow{\sim} V'$ diff. ($V$ nbd. of $\bar{x}$, $V' \subset \mathbb{R}^2$ open)

note $h(C_1 \cap V) \subset \{0\} \times \mathbb{R}$

$h : \underset{(x_1,x_2)}{V} \longrightarrow \underset{(y_1,y_2)}{V'} \qquad y_1 = w(x)$

Consider

$$g = f \circ h^{-1} : V' \longrightarrow \mathbb{R} \qquad V' \subset \mathbb{R}^2$$

Let $\bar{g} = g\big|_{y_1=0} : (\{0\} \times \mathbb{R}) \cap V' \longrightarrow \mathbb{R}$.

The set of crit. values of $\bar{g}$ has measure 0 in $\mathbb{R}$ (previous case)

But each $y \in h(C_1 \cap V)$ is a crit. pt. of $\bar{g}$ :

$$\frac{\partial \bar{g}}{\partial y_2}(y) = \frac{\partial g}{\partial y_2}(y) = \frac{\partial}{\partial y_2}(f \circ h^{-1})(y) = 0$$ since $h^{-1}(y) \in C_1 \cap V$ is a crit. pt for $f$.

Thus $\bar{g}(h(C_1 \cap V))$ has measure 0,

or $f(C_1 \cap V)$ has measure 0. ($V$: nbd of a pt. in $C_1 \setminus C_2$)

Now cover $C_1 \setminus C_2$ by countably many such $V$ to see

meas. $(f(C_1 \setminus C_2)) = 0$.