Existence of partitions of unity

Goal M - C^k manifold

$\ell = \{U_\alpha\}_{\alpha \in A}$ open covering.

want $\{\varphi_\alpha\}_{\alpha \in A}$ $\varphi_\alpha \in C^k(M)$ w/ cpt. support $\subset U_\alpha$

" strictly subordinate " to ℓ

$$\sum_{\alpha \in A} \varphi_\alpha \equiv 1 \text{ on } M.$$

this is ~~there~~ **not** possible in gen'l example $M = \mathbb{R}$

$\ell = \{\mathbb{R}\}$ $\{$ spt $\varphi \in R$ cpt
$\varphi \equiv 1$ on $\mathbb{R}$
(not compatible!)

p.u. $\{\varphi_\beta\}_{\beta \in B}$ **subordinate** to ℓ

if $\{\text{spt } \varphi_\beta\}_\beta$ is a refinement of ℓ

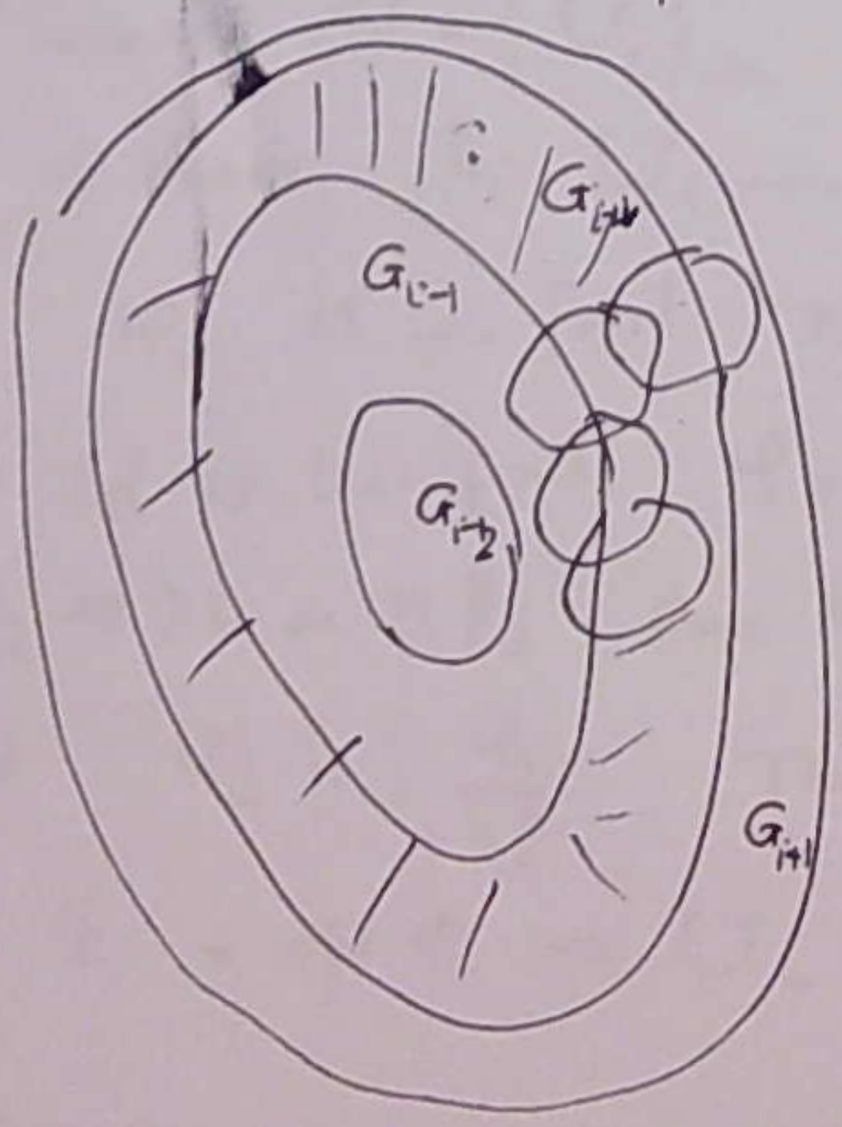
/ $\forall \beta \ \exists \alpha$ s.t. spt $\varphi_\beta \subset U_\alpha$

also: locally finite

(so $\sum_{\beta \in B} \varphi_\beta$ is defined)

recall

X : top. mfld. $\Rightarrow$ X paracompact (proof in 2/5 notes)



$\overline{G_i} \setminus G_{i-1}$ compact

contained in the open

$G_{i+1} \setminus \overline{G_{i-2}}$

construction of p.u.

" bump function in $\mathbb{R}^m$

$\varphi : B(3) \longrightarrow [0,1]$

$\varphi \equiv 1$ on $B(1)$

$\varphi : \mathbb{R}^n \xrightarrow{C^\infty} [0,1]$

$\begin{cases} \varphi \equiv 1 \text{ on } B(1) \\ \varphi \equiv 0 \text{ if } \|x\| \geq 2 \end{cases}$

Given a local chart h on M^n

$\text{spt}\, \varphi = \overline{B}(2) \subset B(3)$

$h : U \longrightarrow B(3) \subset \mathbb{R}^n \qquad h(U) = B(3)$

sets $W \subset V \subset U$

$\downarrow h \quad \downarrow h \quad \downarrow h$

preimages under h $\quad B(1) \subset B(2) \subset B(3)$

define $\varphi_h = \begin{cases} \varphi \circ h \in C^\infty_0(U) \subset C^k(M) & \text{spt } \varphi_h \subset U \\ 0 \quad M \setminus U \end{cases}$

Prop Given $\mathcal{C}$: open cover of M, $\exists$ a C^k p.u. $\{\psi_i\}$ $\psi_i \in C^k(M)$

s.t. $\sum \psi_i \equiv 1$, subordinate to $\mathcal{C}$: spt ψ_i compact; contained in some $U_\alpha \in \mathcal{C}$

Proof Consider $\mathcal{U} = \{U_i\}_{i \geq 1}$ countable, loc. finite open cover by domains of charts $h_i : U_i \longrightarrow B^m(3)$ refining $\mathcal{C}$.

$\varphi_i : M \longrightarrow [0,1]$ spt $\varphi_i = \overline{V_i} \subset U_i$, cpct.

since $\mathcal{U}$ is loc. finite, $\varphi = \sum_{i=1}^{\infty} \varphi_i$ is def. and C^k on M.

(and $\sum \varphi_i > 0$ on M, since the W_i cover M and $\varphi_i \equiv 1$ on W_i.)

Set $\psi_i = \dfrac{\varphi_i}{\varphi}$. Then $\sum_i \psi_i \equiv 1$ on M

and spt ψ_i = spt $\varphi_i \subset U_i$ compact.

Thm Let $\mathcal{C} = (C_\alpha)_{\alpha \in A}$ be an open cover of M.

Then $\exists$ $(\varphi_\alpha)_{\alpha \in A}$ str. subordinate p.u. (same index set)

i.e spt $\varphi_\alpha \subset C_\alpha$ $\forall \alpha \in A$. , $\sum_\alpha \varphi_\alpha \equiv 1$ on M

(& but spt φ_α may no fail to be compact). $\varphi_\alpha \in C^k(M)$.

Proof Consider the ctble, loc. finite cover of M $\mathcal{U} = \{U_i\}_{i \geq 1}$ subordinate to $\mathcal{C}$ as before ($h_i : U_i \longrightarrow \mathbb{B}(3)$)

each $U_i \subset C_\alpha$ for some α

Let $f : \mathbb{N} \longrightarrow A$ be a "choice fn" : $f(i) = \alpha \in A$ i.e., a choice of α s.t $U_i \subset C_\alpha$

Let $\{\psi_i\}$ be the p.u. assoc. to $\mathcal{U}$

And set $\psi_\alpha = \sum_{f(i) = \alpha} \psi_i$ since $(U_i)_{i \geq 1}$ is loc. finite

(*) note spt $\psi_\alpha = \overline{\bigcup_{f(i)=\alpha} \text{spt } \psi_i} = \bigcup_{f(i)=\alpha} \overline{V}_i$ [not nec compact!]

claim $\{\text{spt } \psi_\alpha\}_{\alpha \in A}$ is loc. finite.

pf: Given any $p \in M$ $\exists$ A a nbd. of p s.t. $U_i \cap A \neq \emptyset$ only for finitely many i (say for $i \in J$, J finite $\subset \mathbb{N}$)

Let $A_0 = f(J) \subset A$

If spt $\psi_\alpha \cap A \neq \emptyset$, then $V_i \cap A \neq \emptyset$ for some i s.t. $f(i) = \alpha$

thus $\alpha \in A_0$, a finite set.

Let $\psi = \sum_{\alpha \in A} \psi_\alpha \in C^k(M)$

($\psi > 0$ on M) , let $\varphi_\alpha = \frac{\psi_\alpha}{\psi}$ Then $\sum_{\alpha \in A} \varphi_\alpha \equiv 1$

spt $\varphi_\alpha \subset C_\alpha$ (from *).

Application 1 : differentiable Urysohn's lemma.

F, G closed, disjoint, nonempty subsets of M

Then $\exists\ f \in C^k(M; [0,1])$ s.t. $f|_F \equiv 0$, $f|_G \equiv 1$.

Proof: $\{M \setminus F, M \setminus G\}$ open cover of M

$\left(F \cap G = \emptyset \implies F^c \cup G^c = M. \right)$

Let $\{\psi_F, \psi_G\}$ be a str. subordinate p.u.

$\psi_f + \psi_G \equiv 1$ on M. , supp $\psi_F \subset M \setminus F$
and $\psi_G \subset M \setminus G$

on G $\psi_G = 0$, so $\psi_f \equiv 1$

on F $\psi_F = 0$, so $\psi_G = 1$

So let $f = \psi_F$. Then $f \equiv 0$ on F and $f = 1$ on G □