

Transversality [cont'd] / Paracompactness.

(last time)

- map $\bar{\wedge}$ submanifold $f: M \rightarrow N, S \subset N$

$$\underline{\text{Def}} \quad df(p)[T_p M] + T_q S = T_q N \quad f(p) = q \quad (f \bar{\wedge} S)$$

(1) $f \bar{\wedge} S$ ~~iff~~ equiv to 0_{n-s} is a reg. value of $\pi \circ \psi \circ f$

$(\psi: V_q \rightarrow \mathbb{R}^s \times \mathbb{R}^{n-s} \text{ submfld chart})$

(2) $f \bar{\wedge} S \Rightarrow f^{-1}(S)$ is a submanifold of M

$$\text{codim}_M f^{-1}(S) = \text{codim}_N S, \quad T_p f^{-1}(S) = \text{ker } df(p)^{-1}[T_q S]$$

- $S_1 \bar{\wedge} S_2$ at pts of $S_1 \cap S_2$ ($S_1, S_2 \subset N$)

$$\underline{\text{Def}} \quad T_q S_1 + T_q S_2 = T_q N \quad q \in S_1 \cap S_2$$

$\Leftrightarrow i^* T_q S_2, i: S_1 \hookrightarrow N$ (inclusion)

(i) $S_1 \cap S_2$ is a submanifold of N

$$\text{codim}(S_1 \cap S_2) = \text{codim}(S_1) + \text{codim}(S_2) \quad T_q(S_1 \cap S_2) = T_q S_1 \cap T_q S_2$$

(special case: $T_q S_1 \cap T_q S_2 = \{0\}$; $T_q S_1 \oplus T_q S_2 = T_q N$

$$\Rightarrow \dim(S_1 \cap S_2) = 0 \quad (\text{union of pts.}).$$

$$(3) \quad f_1: M \rightarrow P$$

$$f_2: N \rightarrow P$$

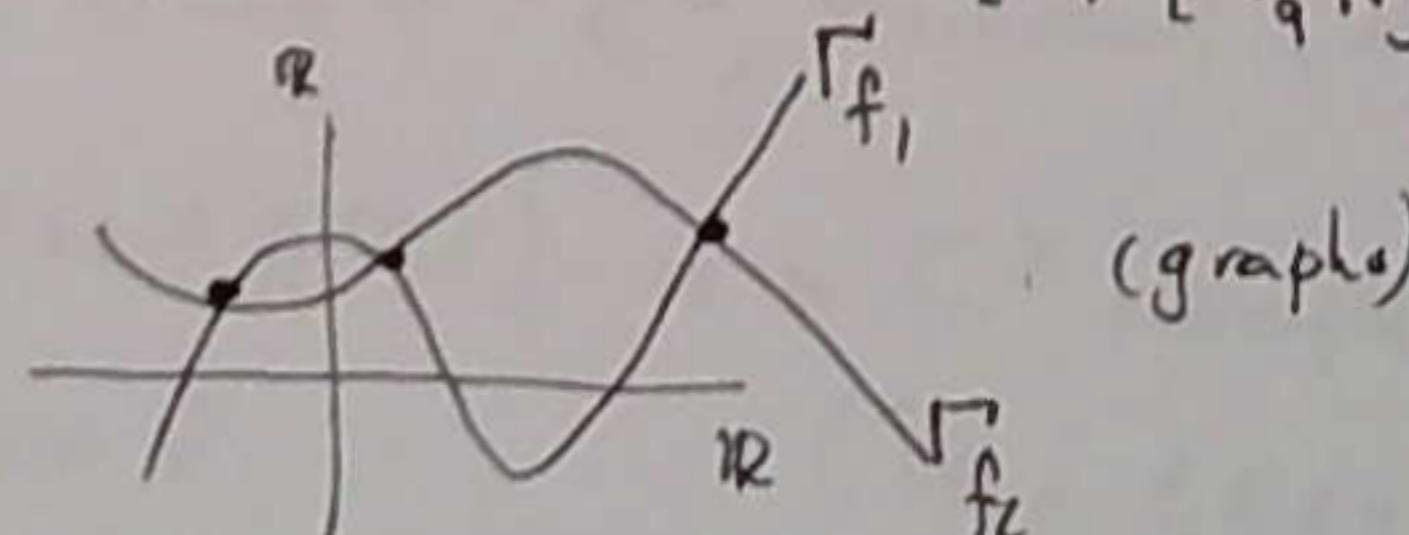
Def. $f_1 \bar{\wedge} f_2$ (at $p \in M, q \in N$ s.t.
 $f_1(p) = f_2(q) = r \in P$)
 if

$$df_1(p)[T_p M] + df_2(q)[T_q N] = T_r P$$

Rk.

$$f_1: \mathbb{R} \rightarrow \mathbb{R}$$

$$f_2: \mathbb{R} \rightarrow \mathbb{R}$$



Exa cse compare $f_1 \bar{\wedge} f_2$ w/ $\Gamma_{f_1} \bar{\wedge} \Gamma_{f_2}$ (in \mathbb{R}^2)

(2)

reduction to transv. of map to submfld

$$(f_1 \times f_2) : M \times N \longrightarrow P \times P$$

$$(f_1 \times f_2)(p, q) = (f_1(p), f_2(q))$$

Prop. $f_1 \pitchfork f_2$ at p, q ($f(p) = f(q) = r$)

$$\Delta \subset P \times P \quad \begin{array}{l} \text{diagonal} \\ (\text{submfld.}) \end{array}$$

$$\{(r, r) ; r \in P\}$$

$$(f_1 \times f_2) \pitchfork_{(p, q)} \Delta \quad \text{at } (r, r) \in \Delta$$

Lm. Alg lemma

E : v.sp., $A, B \subset E$ subspaces.

$$\text{Then } A + B = E \iff (A \times B) + D = E \times E$$

$$E \times E : \left\{ \begin{array}{l} (e_1, e_2) + (f_1, f_2) = (e_1 + f_1, e_2 + f_2) \\ \lambda(e_1, e_2) = (\lambda e_1, \lambda e_2) \end{array} \right. \quad (\text{diff from } E \otimes E)$$

$A \times B$ subsp. of $E \times E$, $D = \{(e, e) \in E \times E ; e \in E\}$ subspace

Prop. follows let by $A = T_p M$, $B = T_q N$, $E = T_r P$

Examples

① Any $f: M \rightarrow N$ is transv. to the identity $\text{id}: N \rightarrow N$
(since the identity is a submersion: submersions are transv. to any map)

Thus the graph of $f: \Gamma_f = \{(p, q) \in M \times N ; q = f(p)\}$

is a submanifold of $M \times N$, of dimension $= \dim M$.

Prop. $f: M \rightarrow P$, $g: N \rightarrow P$ C^k transversal (at all points w/
same mfd)

$\Rightarrow Q = \left\{ (p, q) \in M \times N \mid f(p) = g(q) \right\}$ is a C^k submfld of $P \times Q$,
of codimension (Q) $= \dim P$. ($Q = (f \times g)^{-1}(\Delta)$)

(2) V : v.sp. $A \in L(V)$ $\Delta \subset V \times V$ diag. (subsp ∞)
 $\Gamma_A = \{(v, Av) \in V \times V\}$ graph (subsp ∞).

$\Gamma_A \cap \Delta \iff \lambda$ is not an eigenvalue of A (exercise in GP)

(3) X : vector field on M (section of $TM \xrightarrow{\pi} M$)

singularity of X : $p \in M$ s.t. $X(p) = 0$

Prop. p is a simple singularity if $\text{rank } dX(p) = m$ ($X: M \rightarrow TM$)
Exercise All sing'ls of X are simple $\iff X \pitchfork \sum_0$ $\pi \circ X = id_M$

\sum_0 : zero section of TM $\Sigma_0 = \{(p, 0_p); 0_p \in T_p M\}$

Difffble partitions of unity.

Def. (paracompactness). M : top. space (Hausdorff).

(1) A collection $\{A_\alpha\}$ of subsets of M is locally finite if $\forall p \in M \exists U_p$ s.t. U_p intersects only finitely many A_α .

(2) M is paracompact if any open cover $\{U_\beta\}_{\beta \in B}$ of M admits a locally finite refinement $\{U_\beta\}_{\beta \in B} \cup U_\beta = M$, each $U_\beta \subset A_\alpha$ for some α .

Rk. • Paracompact Hausdorff spaces are normal (see [Munkres])
• Metrizable spaces are paracompact ("Michel's thm")

Prof: X : loc. compact Hausdorff, 2nd countable.

Then. (i) X is σ -compact.

(ii). X is paracompact (each set of the refinement can be taken precompact).

(4)

(1) \underline{X} is σ -compact:

Let $\{U_i\}_{i \geq 1}$ otble basis of X , U_i precompact open sets.

Let want: $X = \bigcup_{i \geq 1} G_i$ G_i open, \overline{G}_i compact
 $\overline{G}_i \subset G_{i+1} \quad i \geq 1$.

Let $G_1 = U_1$. (\overline{G}_1 cpt.)

Given $G_k = U_1 \cup U_2 \cup \dots \cup U_{j_k}$ let J_{k+1} be smallest s.t.
 $\overline{G}_k \subset \bigcup_{i=1}^{j_{k+1}} U_i \quad \overline{G}_k$ cpt.

Set $G_{k+1} = \bigcup_{i=1}^{j_{k+1}} U_i$

so X is σ -compact.

(2) \underline{X} is paracompact

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover.

4 $i \geq 3$ $\overline{G}_i \setminus G_{i-1}$ is cpt, contained in the open $G_{i+1} \setminus \overline{G}_{i-2}$.

choose a finite subcover of the open cover

$\{U_\alpha \cap (G_{i+1} \setminus \overline{G}_{i-2})\}_{\alpha \in A}$ of $\overline{G}_i \setminus G_{i-1}$

Then choose a finite subcover of the open cover $\{U_\alpha \cap G_3\}$ of \overline{G}_2 .

This collection of open sets is a otble, loc. finite refinement of $\{U_\alpha\}$, consisting of precompact open sets. \square

Notation Given a chart (U, h) for M , $h: U \rightarrow B(3) \subset \mathbb{R}^m$,

let $W \subset V \subset U$ be: $V = h^{-1}(B(2))$,
 $W = h^{-1}(B(1))$.

Prop C : open cover of M

Then \mathcal{C} has a countable, locally finite refinement $\{U_1, U_2, \dots\}$ domains of charts $h_i : U_i \rightarrow B(3) \subset \mathbb{R}^m$, s.t. the corresponding W_i are a (loc. finite) cover of M .

Proof Let $M = \bigcup_{i \geq 1} G_i$ as before : $\overline{G_i} \subset G_{i+1}$, G_i open, $\overline{G_i}$ compact

G_2 has a finite cover by open sets of type W , whose const. U are contained in G_3 and in some set of \mathcal{C} .

$\overline{G_3} \setminus G_2$ (compact) has a finite cover by open sets of type W , s.t. each of the const. U is contained in $G_4 \setminus \overline{G_1}$ and in some open set of \mathcal{C} .

continue

We get countable covers $W = \{W_1, W_2, \dots\}$, $U = \{U_1, U_2, \dots\}$ where U refines C and is loc. finite (since each U_i is contained in some $\overline{G_j}$ (cpt) and thus intersects only finitely many other U').

Remark When M is compact, the prop. is trivial
(U may be taken finite.)