

Recall

$f: M \xrightarrow{C^r} N$ diff'ble

Def. $q \in N$ is a regular value if $d_f(p)$ is onto, $\forall p \in f^{-1}(q)$

Thm. q reg. value $\Rightarrow f^{-1}(q) = L_q$ is a submanifold of M of codimension $= \dim N$

$$T_p L_q = \ker(d_f(p)) \subset T_p M$$

Generalization (due to R. Thom, 1956s)

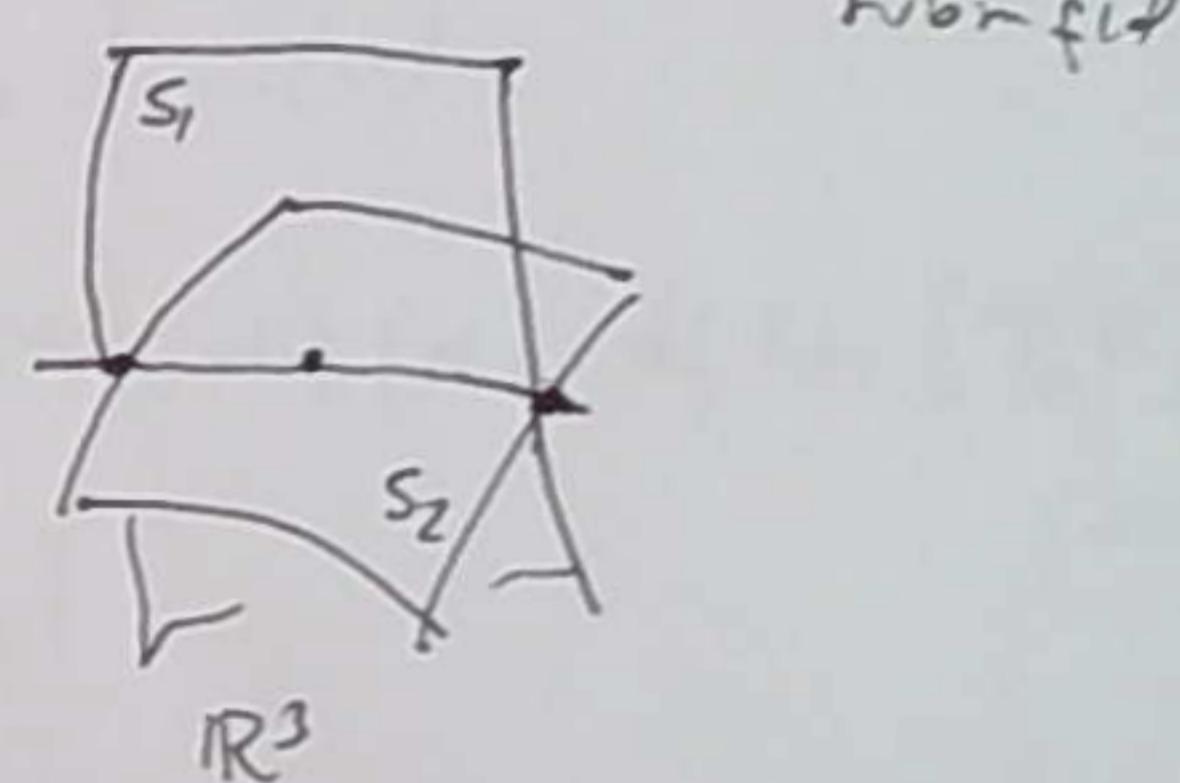
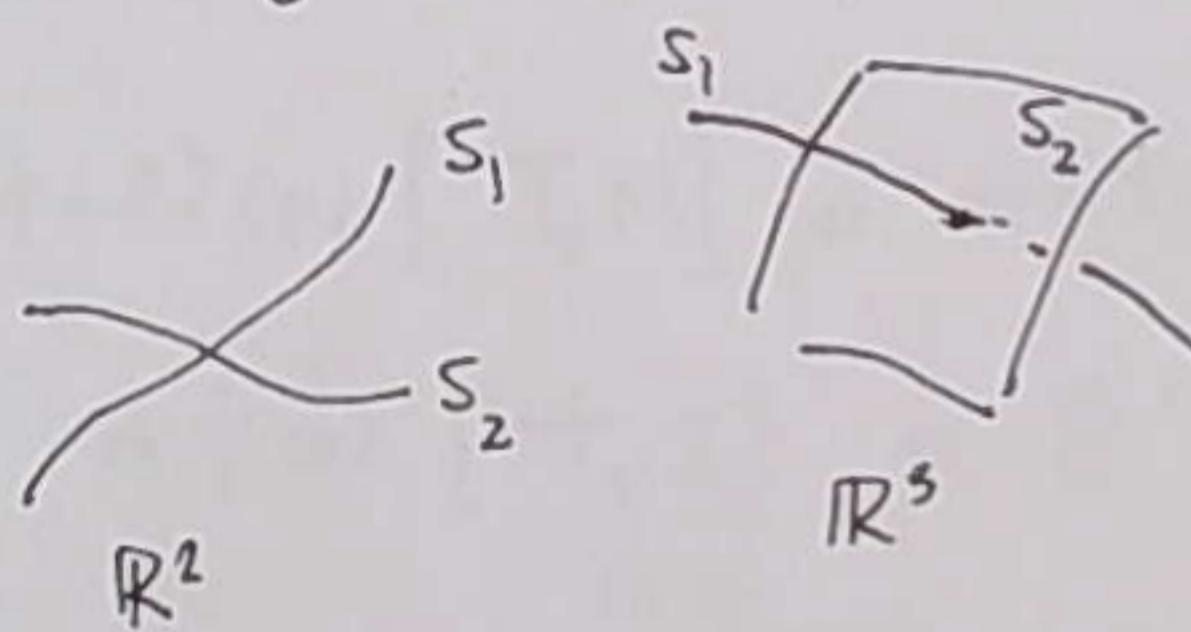
$$d_f(p) \in L(T_p M, T_{f(p)} N).$$

Transversality

- of submanifolds $S_1 \pitchfork S_2$

main \rightarrow map to submanifold $f \pitchfork S$

- $f \pitchfork g$ (two maps)

Examples

Def $f: M \xrightarrow{C^r} N$ $S \subset N$ submanifold $q \in S$

f is transv. to S at q if $f(p) = q$ $p \in M$ ($f \pitchfork_s S$)

and $d_f(p)[T_p M] + T_q S = T_q N$ (not always a direct sum!)

($S_1 \pitchfork S_2$ transv. if $i \pitchfork_j S_2$ $i: S_1 \rightarrow N$ inclusion)

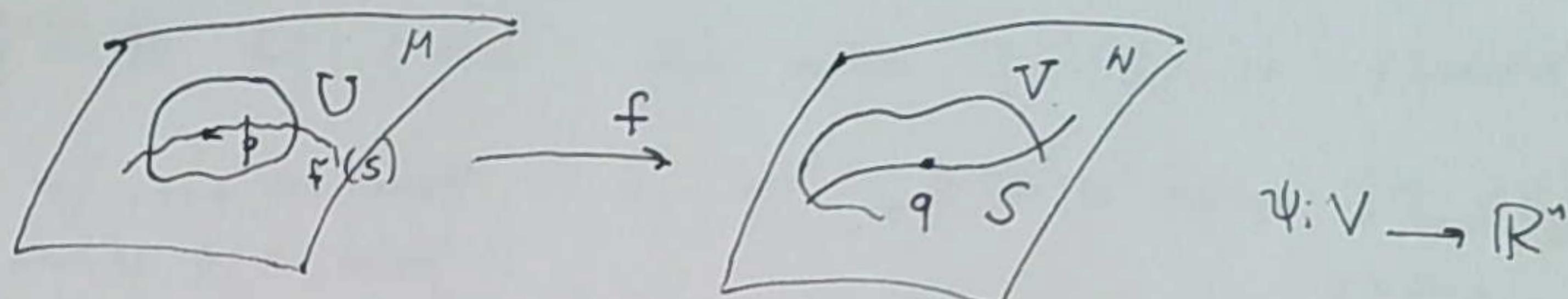
(ex. $S = c \in N$ $f \pitchfork_s c \Leftrightarrow d_f(p)$ is onto)

Main Theorems

1- "transversality may be reduced to regular value."

2- If $f \pitchfork S$ at each $p \in f^{-1}(S)$, then

$f^{-1}(S)$ is a submanifold of M . $\begin{cases} \text{codim}(f^{-1}(S)) = \text{codim } S \\ T_p f^{-1}(S) = df^{-1}(p)[T_q S] \end{cases}$

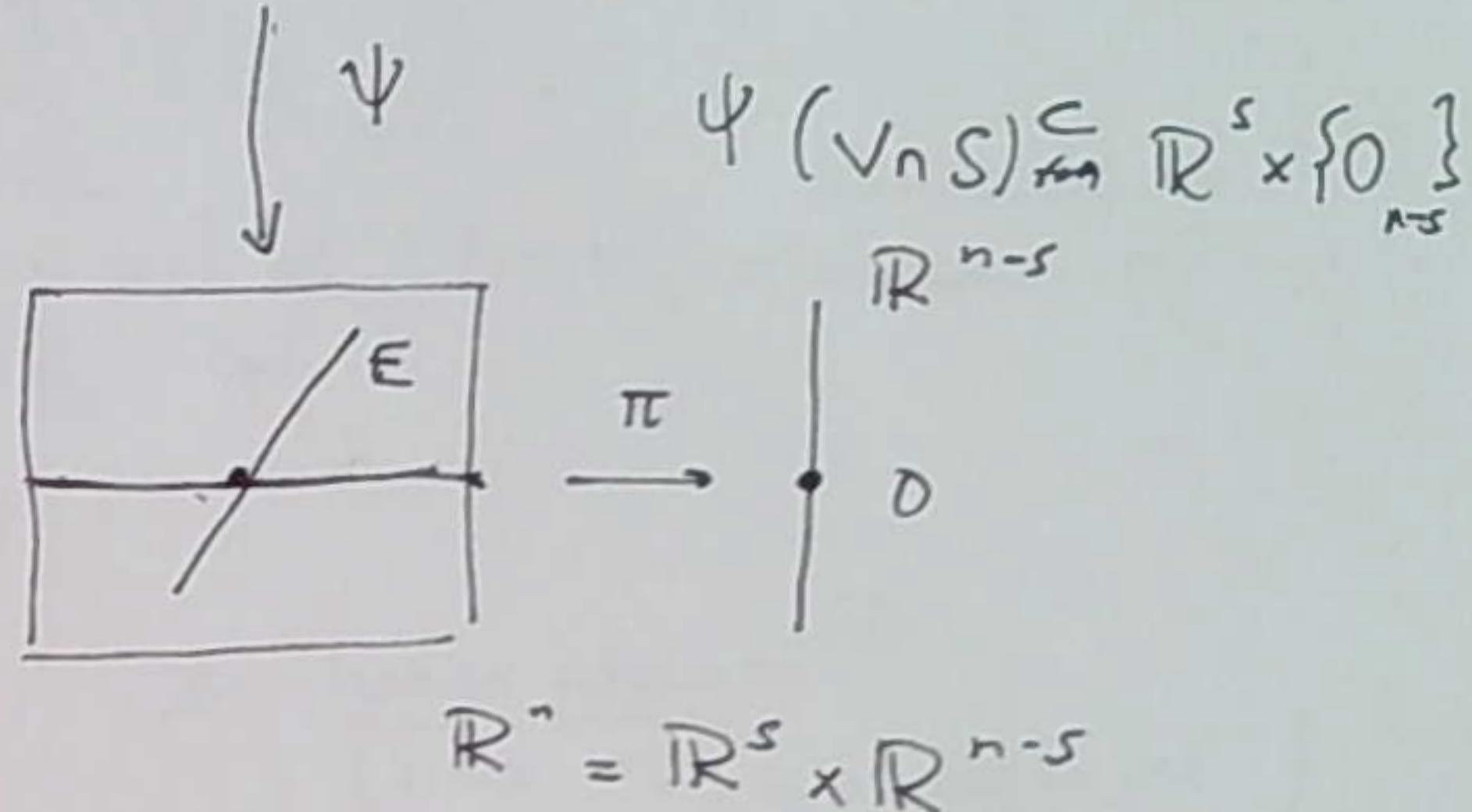
Prop

$$U \subset M \text{ nbhd of } p \text{ in } M \\ f(U) = S$$

f is transv. to S at points of U .

iff 0_{n-s} is a reg. value of

$$\pi \circ \psi \circ f|_U: U \rightarrow \mathbb{R}^{n-s}.$$



$$D = \mathbb{R}^s \times \mathbb{R}^{n-s}$$

Pf. Let $p \in U \cap f^{-1}(S) = (\pi \circ \psi \circ f|_U)^{-1}(0)$

$$d(\psi \circ f)(p)[T_p M] = d\psi(q)[T_q N] \quad d\psi(q) \cdot df(p)[T_p M] = E \subset \mathbb{R}^s$$

$$d\psi(q)[T_q S] = \mathbb{R}^s \times \{0_{n-s}\} \quad (\text{subspace})$$

Since $d\psi(q)$ is an isomorphism TFAE

$$df(p)[T_p M] + T_q S = T_q N \quad (f \text{ transv. } S)$$

$$E + \mathbb{R}^s \times \{0\} = \mathbb{R}^s \times \mathbb{R}^{n-s}.$$

$$d\pi(E) = \mathbb{R}^{n-s}$$

$$d(\pi \circ \psi \circ f)(p)[T_p M] = d\pi \circ \overbrace{d\psi \circ df}[T_p M] = d\pi[E] = \mathbb{R}^{n-s}$$

i.e. $d(\pi \circ \psi \circ f)(q)$ is onto ($\Leftrightarrow 0$ is a reg. value of $\pi \circ \psi \circ f|_U$)

As c wnsq.

at every $p \in f^{-1}(S)$ (assumed nonempty).

Prop ~~f~~ If $f: S \rightarrow M$, then either $f^{-1}(S)$ is a submanifold

of M w/ same codimension as S ; and $T_p f^{-1}(S) = df(p)^{-1}[T_q S]$

Pf U : neighborhood of p as before.

$$f^{-1}(S) \cap U = \underbrace{(\pi \circ \psi \circ f|_U)^{-1}(0_{n-s})}_{\text{submanifold of } U \subset M, \text{ of dimension } n-s}$$

$p \in U$

$$\begin{aligned} T_p f^{-1}(S) &= \ker d(\pi \circ \psi \circ f|_U)(p) \\ &= df(p)^{-1}(T_q S). \end{aligned}$$

(since $T_q S = \ker d(\pi \circ \psi)$ & linear algebra \oplus
(since ψ is a diff., $d\psi(q)$ is an isom.)

Cor. Let N^s, S^r be submanifolds of M^m ^{each} intersecting transversely at $p \in N \cap S$. $N \cap S \subset M$

i.e.
(def.) $T_p N + T_p S = T_p M$

Then $N \cap S$ is a submanifold of M .

After the prop. to w/ dimension $n+s-m$

($i: N \hookrightarrow M$ inclusion. $N \cap S = i^{-1}(S)$ $i \bar{\rightarrow} S$ (from hypothesis)
 $n - \dim(N \cap S) = \text{codim}_N(N \cap S) = \text{codim}_M(S) = m-s$)

$$T_p(N \cap S) = i^{-1}[T_p S] = T_p N \cap T_p S.$$

\oplus Exercise

(i) Let $T: V \rightarrow U$, $S: U \rightarrow W$ be linear maps.

Then $\ker(ST) = T^{-1}(\ker(S)) \subset U$ (preimage under T)

(ii) Explain why this implies (*):

$$\ker d(\pi \circ \psi \circ f|_U)(p) = df(p)^{-1}(T_q S)$$

Next: transversality of maps.