

C¹ Whitney topology

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(M, g) , $N \subset \mathbb{R}^N$ surface of class C^k

$W^1(M, N) : C^1$ maps $f : M \rightarrow N$, w/ basis $\tilde{W}^1(f; \varepsilon)$

(at f) $\varepsilon : M \rightarrow \mathbb{R}_+$ cont.

in \mathbb{R}^N

$L(TM; \mathbb{R}^N)$

↓ using g

$$W^1(f; \varepsilon) = \{g \in W^1(M, N); |f(p) - g(p)| < \underbrace{\varepsilon(p)}, |\mathrm{d}f(p) - \mathrm{d}g(p)| < \varepsilon(p)\}$$

Rk. M cpt $\rightarrow \varepsilon(p) \geq \varepsilon_0 > 0$ on M

$$\|f - g\|_1 < \varepsilon$$

$W^1(M, N) \approx C^1(M, N)$ (uniform C^1 topology)

$$|f - g| + \|\mathrm{d}f - \mathrm{d}g\| < \varepsilon_0 \text{ on } M$$

topology

invariance; openness of immersions, submersions.

Invariance

A) $\varphi : M_1 \rightarrow M_2$ induces

$\varphi^* : W^1(M_2, N) \rightarrow W^1(M_1, N)$

$$\varphi^*(f) = f \circ \varphi$$

[Prop.] cont. if φ is proper

Appln. top. on $W^1(M, N)$
indep. of g on M

B) $\varphi : N_1 \rightarrow N_2$ induces

$\varphi_* : W^1(M, N_1) \rightarrow W^1(M, N_2)$

$$\varphi_* f = f \circ \varphi$$

[Prop.] cont. in general.

(2)

goal proving (continuity) invariance for $\varphi^*: W'(M_2, N) \rightarrow W'(M_1, N)$

Prop. 1

$\varphi: M_1 \rightarrow M_2 \in C^1$ map.

given $K \subset M_1$ compact, $\gamma > 0$.

Then $\exists \delta > 0$ s.t. $f, g \in W'(M_2, N)$,

$$\|f - g\|_1 < \delta \text{ on } \varphi(K) \Rightarrow \|f \circ \varphi - g \circ \varphi\|_1 < \gamma \text{ on } K.$$

(this is continuity of φ^* if M_1 is compact).

Proof

$$|f(\varphi(p)) - g(\varphi(p))| < \delta \text{ for } p \in K \quad |f(q) - g(q)| < \delta \quad q \in \varphi(K)$$

(clear).

Chain rule

$$d(f \circ \varphi) - d(g \circ \varphi) = (df \circ \varphi - dg \circ \varphi)[d\varphi]$$

Let $A \geq 1$ be s.t. $\|d\varphi(p)\| \leq A$, $p \in K$ ($\varphi \in C^1(M_1, M_2)$)

$$\text{Let } \delta = \frac{\gamma}{A}.$$

$$|d(f \circ \varphi) - d(g \circ \varphi)|_K \leq \underbrace{\|df \circ \varphi - dg \circ \varphi\|}_K \underbrace{\|d\varphi\|}_K \leq \delta A = \gamma$$

$< \delta$ since $\|f - g\|_1 < \delta$ on $\varphi(K)$

$$\therefore \|f \circ \varphi - g \circ \varphi\|_{C^1(K)} < \gamma.$$

Cor. 1

continuity of φ^* in $W'(M_2, N)$. $\varphi^*: W'(M_2, N) \rightarrow W'(M_1, N)$

want. if $f \in W'(M_2, N)$, $\varepsilon: M_1 \rightarrow \mathbb{R}_+$ cont.

then $\exists \delta: M_2 \rightarrow \mathbb{R}_+$ so that

$$g \in W'(f, \delta) \Rightarrow g \circ \varphi \in W'(f \circ \varphi, \varepsilon) \subset W'(M_1, N)$$

ε has a ~~con~~ fixed lower bound $\varepsilon > \varepsilon_0$ on M_1 (M_1 compact)

The Prop. gives constant $\delta > 0$ s.t.

$$\in C^1(f \circ \varphi, \varepsilon_0)$$

$$g \in W'(f, \delta) \Rightarrow g \circ \varphi \in W'(f \circ \varphi, \varepsilon_0) \subset W'(f \circ \varphi, \varepsilon)$$

$= C^1(f, \delta)$

QED \square

Extension to proper maps

Prop: $\varphi: M_1 \rightarrow M_2$ C^1 proper map.

Then $\varphi^*: W'(M_2; N) \rightarrow W'(M_1; N)$ $\ell^*(f) = f \circ \varphi$
is cont.

[Lem.] $\gamma: M_1 \rightarrow \mathbb{R}_+$ pos. cont.

and $\varphi: M_1 \rightarrow M_2$ cont., proper.

Then \exists a pos. cont. fn. $\delta: M_2 \rightarrow \mathbb{R}_+^*$ s.t.
 $0 < \delta \circ \varphi < \gamma$ on M_1 .

follows from (last time)

$\{\mathcal{E}_\alpha\}_{\alpha \in A}$ loc. finite cover of M_2 (top. mfd).

$\{\alpha_\alpha\}_{\alpha \in A}$ pos. real numbers.

$\Rightarrow \exists \delta: M_1 \rightarrow \mathbb{R}_+$ cont. s.t. $0 < \delta(p) < \alpha_\alpha$, $p \in \mathcal{E}_\alpha$.

Pf of lemma

Let $\{K_\alpha\}_{\alpha \in A}$ be a loc-finite cover of M_2 by compact sets.

φ proper $\Rightarrow \varphi^{-1}(K_\alpha) \subset M_1$ compact so $\inf \{\gamma(p); p \in \varphi^{-1}(K_\alpha)\} = \alpha_\alpha$
(i.e. $\varphi^{-1}(K_\alpha) = \emptyset$, set $\alpha_\alpha = 1$).

so $\exists \delta: M_2 \rightarrow \mathbb{R}_+$ cont. s.t. $0 < \delta(q) < \alpha_\alpha$, $q \in K_\alpha$

If $p \in M_1$, then $\varphi(p) \in K_\alpha$ (for some $\alpha \in A$) $\Rightarrow \delta(\varphi(p)) < \alpha_\alpha \leq \gamma(p)$.
i.e. $0 < \delta \circ \varphi < \gamma$ on M_1 .

next. $\varphi_*: W'(M, N_1) \rightarrow W'(M, N_2)$ is cont.

$\varphi_*(f) = f \circ \varphi$, $\varphi: N_1 \rightarrow N_2$. (see notes)

$(N_1, N_2 \subset \mathbb{R}^n)$ surfaces.

Openness of immersions

Lemr $U \subset \mathbb{R}^m$ open, $K \subset U$ compact.

$f: C^1(U; \mathbb{R}^n)$ immersion

Then $\exists \gamma > 0$ st. $g \in C^1(U; \mathbb{R}^n)$, $\|g - f\|_{C^1(K)} < \gamma$

$\Rightarrow g|_K$ is an immersion.

Pf $\mathcal{O} = L(\mathbb{R}^m; \mathbb{R}^n) = L$

' set of lin. maps w/ trivial kernel.

$df: U \rightarrow L$ maps the compact set K to \mathcal{O}

so $Im(df)$ is a compact set in L closed in L

Let $\gamma = \text{dist} [df(K), L \setminus \mathcal{O}] > 0$

/ \uparrow compact subset of \mathcal{O}

in a norm

if $g \in C^1(U; \mathbb{R}^n)$, $\|dg(p) - df(p)\| < \gamma$, $p \in K \Rightarrow dg \in \mathcal{O}$

so $g|_K$ is an immersion.

Proof. The C^1 immersions define an open subset of $W^1(M, N)$

Pf. $M = \bigcup_{i \geq 1} U_i$ be a cov. of M by domains of chart,

w/ $V_i = \overline{V_i} \subset U_i$ ($\overline{V_i}$ compact, V_i cover M).

Let $f \in W^1(M, N)$ be an immersion.

from the lemma { \exists a seq. (a_i) of pos. numbers s.t.

if $g \in W^1(M, N)$, $\|g - f\|_{C^1(\overline{V_i})} < a_i \Rightarrow g|_{\overline{V_i}}$

so if $g \in W^1(f; \tilde{\alpha})$ then g is an immersion.

an immersion

$(\tilde{\alpha}_i)_{i \geq 1}$

$W^1(f; \tilde{\alpha}) = \{g \in W^1(M, N); \|f - g\|_{C^1(\overline{V_i})} < a_i\} \quad \text{basis for } W^1(M, N)$