

from 1/27 lecture.

1/29 (6)
(from 1/27 lecture)

Prop. $f: M \rightarrow N$ C^k immersion ($k \geq 1$) and injective.

If $f: M \rightarrow f(M)$ is an open map (where $f(M)$ has the topology induced from N)
then $f(M)$ is a submanifold of N .

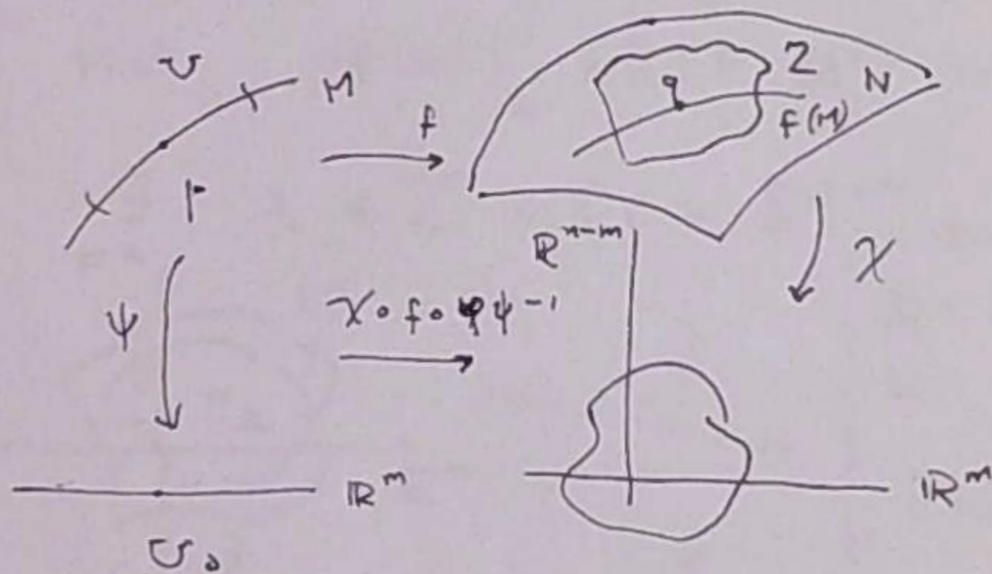
Proof. We need to show that $\forall q \in f(M) \exists (W, \varphi)$ chart for N at q

s.t. φ maps $W \cap f(M)$ to $\mathbb{R}^m \times \{0_{n-m}\} \subset \mathbb{R}^n$.

Let $q \in f(M)$, $q = f(p)$. \exists charts (U, ψ) at p , (Z, χ) at q (for N)

s.t. for some splitting $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$: $(w/ f(U) \subset Z)$

$$\chi \circ f \circ \psi^{-1}(x) = (x, 0_{n-m}) \quad \forall x \in U_0 = \varnothing \psi(U) \subset \mathbb{R}^m$$



Since $f: M \rightarrow f(M)$ is an open map, $f(U)$ is open in $f(M)$.

Thus $f(U) = f(M) \cap W$, where we may assume $W \subset Z$
(~~not~~ or replace W by $W \cap Z$)

So $\chi|_W$ maps $f(M) \cap W$ to $\mathbb{R}^m \times \{0_{n-m}\}$

and we may take $\varphi = \chi|_W$.

Remark From the local form of immersions, it follows that

$f|_U$ is a homeo. from U to $f(U)$ (w/ the ind. topology). But

in general $f(U)$ is not open in $f(M)$ (if we don't assume $f: M \rightarrow f(M)$ is an open map)

Goal: Preimages of regular values are submanifolds

$f: M^m \longrightarrow N^n$ is a submersion if $df(p)$ is surjective
($m \geq n$) $\forall p$

$c \in N$ is a regular value if $f^{-1}(c) = \emptyset$
or $\forall p \in f^{-1}(c)$ $df(p)$ is onto

Local form of submersions (in \mathbb{R}^n)

Thm $U \subset \mathbb{R}^{n+m}$ open $f: U \xrightarrow{C^k} \mathbb{R}^n$

Suppose $z_0 \in U$ is s.t. $df(z_0) \in L(\mathbb{R}^{m+n}; \mathbb{R}^n)$ is onto.

choose a direct sum splitting $\varphi: \mathbb{R}^{m+n} = E^m \oplus F^n$ ($z_0 = (x_0, y_0)$)
so that $df(z_0)|_F \in L(F; \mathbb{R}^n)$ is an isomorphism.

Then

$\exists z_0 \in Z \subset U \subset \mathbb{R}^{n+m}$ open.

and

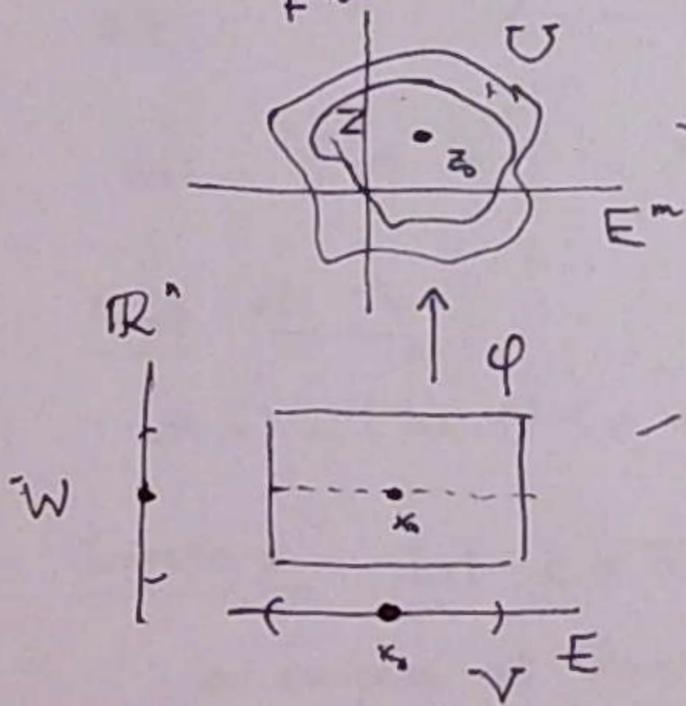
$x_0 \in V \subset E^m$

$f(z_0) \in W \subset \mathbb{R}^n$

and

$\varphi: V \times W \rightarrow Z$

(C^k diff.)



s.t.

$f \circ \varphi: V \times W \longrightarrow W$

has the form $(f \circ \varphi)(x, w) = w$ (projection)

(consequence: submersions are open maps)

Proof idea: reduce to the I.F.T.

Proof Define $h: U \rightarrow E^m \times \mathbb{R}^n$ by
 $\hookrightarrow E^m \times F^n$

$z_0 = (x_0, y_0)$

$h(x, y) = (x, f(x, y))$

$h(z_0) = (x_0, f(z_0))$

Claim at z_0 $dh(z_0) \in L(E^m \times F^n; E^m \times \mathbb{R}^n)$
 is an iso

Proof of claim

$dh(z_0)[u, v] = (u, d_1 f(z_0)[u] + d_2 f(z_0)[v])$

$u \in E, v \in F$

$dh(z_0)[u, v] = 0 \implies u = 0, d_2 f(z_0)[v] = 0 \implies v = 0$

but $d_2 f(z_0) = df(z_0)|_F \in L(F; \mathbb{R}^n) \stackrel{\text{iso}}{=} \mathbb{R}^n$

$\text{Ker } dh(z_0) = 0$
 (proves claim).

By the IFT $\exists Z, V, W$ open

$z_0 \in Z = \mathbb{R}^{n+m}, f(z_0) \in W \subset \mathbb{R}^n, x_0 \in V \subset E$

s.t. $h: Z \rightarrow V \times W$ is a diffeo.

Let $\varphi = h^{-1}: V \times W \rightarrow Z$ claim $(f \circ \varphi)(x, w) = w$
 $\forall (x, w) \in V \times W$

Proof of claim

$(x, w) = (h \circ \varphi)(x, w) = (x, (f \circ \varphi)(x, w)) \implies (f \circ \varphi)(x, w) = w. \quad \square$

Corollary Let $c \in \mathbb{R}^n$ be a regular value of f . Then $f^{-1}(c) \stackrel{M_c}{=} M_c$
 is a surface of class C^k in \mathbb{R}^{n+m} , of dimension m . (If nonempty!)

Let $z_0 \in M_c, V, W, \varphi$ as above

Proof Write $\varphi(x, w) = (x, \varphi_2(x, w))$, where $\varphi_2: V \times W \rightarrow F^n$

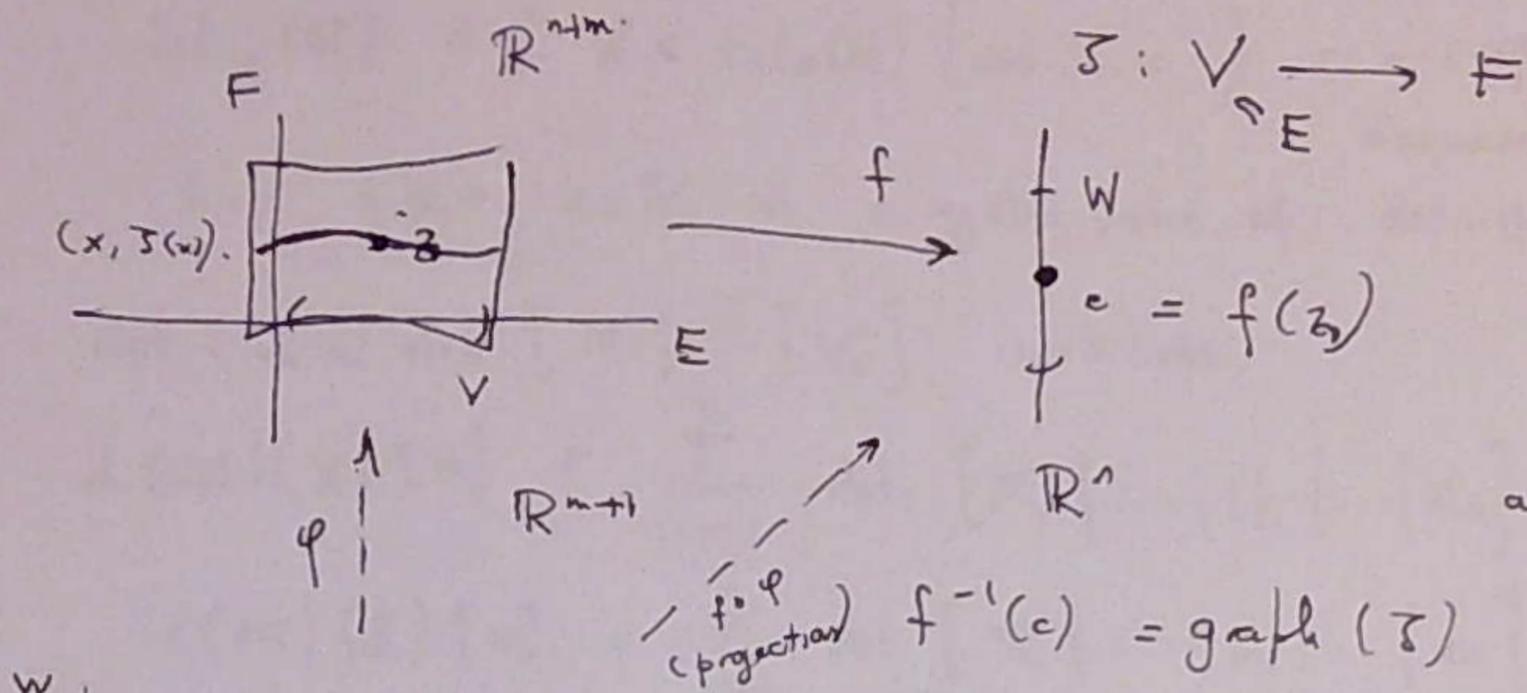
Define $\mathcal{J}(x) = \varphi_2(x, c)$ Then

(i) $f(x, \mathcal{J}(x)) = f(x, \varphi_2(x, c)) = (f \circ \varphi)(x, c) = c$

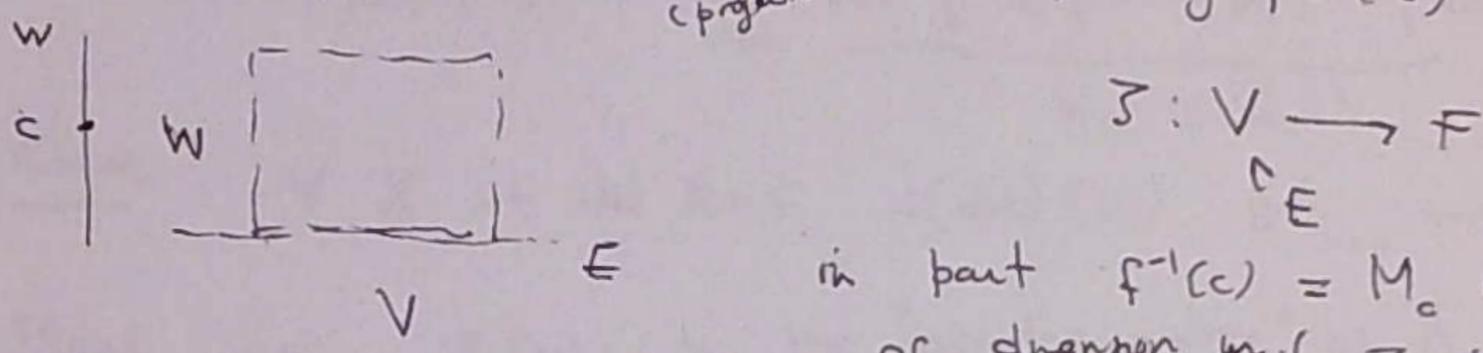
(ii) Conversely if $(x, y) \in Z \subset E \times F$ and $f(x, y) = c$
 we have: $(x, y) = (\varphi \circ h)(x, y) = (x, \varphi_2(x, f(x, y))) = (x, \varphi_2(x, c))$
 $\rightarrow = (x, \varphi_2(h(x, y))) = (x, \mathcal{J}(x))$

Conclusion

$$f^{-1}(c) \cap Z = \text{graph}(\zeta)$$



↳ Implicit fn. thm:
the preimage of a reg. value is locally a graph



Claim $T_{z_0} M_c = \ker(df(z_0))$

Note - (the differential of ζ). $f(x, \zeta(x)) \equiv c \quad \forall x \in V$
 $0 = \partial_1 f(x, \zeta(x)) + \partial_2 f(x, \zeta(x)) d\zeta(x)$

so $d\zeta(x) = -\partial_2 f(x, \zeta(x))^{-1} \partial_1 f(x, \zeta(x)) \in L(E, F)$

check $\partial_1 f(x, \zeta(x)) \in L(E, \mathbb{R}^n)$ $\partial_2 f(x, \zeta(x)) \in L(F, \mathbb{R}^n)$ (iso).
 $\partial_2 f(x, \zeta(x))^{-1} \partial_1 f(x, \zeta(x)) \in L(E, F)$.

pf of claim Let $v \in T_{z_0} M_c$. Let $\alpha: \mathbb{R} \rightarrow M_c$ with $\alpha(0) = z_0, \alpha'(0) = v$.
 $f(\alpha(t)) \equiv c = f(z_0)$.

$df(z_0)[v] = 0, \text{ or } v \in \ker df(z_0)$.

so $T_{z_0} M_c \subseteq \ker df(z_0)$. Both have dimension $m = \dim E$
 (proves claim).

Example $GL_n(\mathbb{R}) \stackrel{\text{open}}{\subset} M_{n \times n}(\mathbb{R})$ $n \times n$ invertible matrices.

$SL_n(\mathbb{R}) = \{ X \in GL_n(\mathbb{R}) \mid \det X = 1 \}$ is a surface of dimension $n^2 - 1$

Any $c \neq 0, c \in \mathbb{R}$ is a regular value of $\det: M_{n \times n} \rightarrow \mathbb{R}$

$\det X = \det [X_1 \mid \dots \mid X_n]$ multilinear

$$d(\det)(X)[H] = \sum_{i=1}^n \det [X_1 \mid \dots \mid H_i \mid \dots \mid X_n]$$

$$d(\det)(I)[H] = \sum_{i=1}^n \det [e_1 \mid \dots \mid H_i \mid \dots \mid e_n] = \sum_i h_{ii} = \text{tr} H.$$

h_{ii} (check).

Claim $\forall X$ s.t. $\det X = 1$. $d(\det)(X): M_{n \times n} \rightarrow \mathbb{R}$ is onto

Proof Let $X(r,s)$ be the "cofactor matrix" of X (delete row r , column s)

$$\det X = \sum_{r,s=1}^n (-1)^{r+s} x_{rs} \det [X(r,s)]$$

$$\frac{\partial}{\partial x_{rs}} (\det)(X) = (-1)^{r+s} \det [X(r,s)]. \quad \left(\text{note } \frac{\partial}{\partial x_{rs}} (\det)(X) = d(\det)(X)[E_{rs}] \right)$$

If $\det X \neq 0$, some $\det [X(r,s)] \neq 0$.

1 at (r,s)
0 elsewhere.

Thus $d(\det)(X)[E_{rs}] \neq 0$.

This shows any $c \neq 0$ is a reg. value of \det .

In particular

$SL_n(\mathbb{R}) = \det^{-1}(1)$ is a surface in \mathbb{R}^{n^2} of dimension $n^2 - 1$

$$T_{\Pi} SL_n(\mathbb{R}) = \ker d(\det)(\Pi) \quad \text{tangent sp. at } \Pi \\ = \{ H \in M_{n \times n} \mid \text{tr} H = 0 \}$$

Note $SL_n(\mathbb{R})$ is a group, but not compact (not a bounded subset of \mathbb{R}^{n^2}).

② Orthogonal group

$$O(n) = \{ X \mid X X^T = I \}$$

$$f: M_{n \times n} \rightarrow M_{n \times n}$$

$$f(X) = X X^T \in S_n \subset M_{n \times n} \quad (\text{vector subspace of dimension } \frac{n(n+1)}{2})$$

$$df(X)[H] = H X^T + X H^T \in S_n$$

Ass. $f(X) = I$

claim $df(X): M_{n \times n} \rightarrow S_n$ is onto.

Let $V \in S_n$, $H = \frac{VX}{2} \in M_{n \times n}$

$$df(X)[H] = \frac{V X X^T}{2} + \cancel{X X^T V} = \frac{V + V^T}{2} = V$$

Thus $O(n)$ is a sfc of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$ in \mathbb{R}^{n^2}

$$T_{\mathbb{I}} O(n) = \ker df(I) = \text{Skew}_n \subset \mathbb{R}^{n^2} \quad (\text{dimension } \frac{n(n-1)}{2})$$

($A^T = -A$)

$O(n)$ is cpt w/ 2 c.c., one of them $SO_n = SL_n \cap O(n)$

(closed & bdd)

$$X = [x_1 \mid \dots \mid x_n] \quad \& \quad \|x_i\| = 1$$

$$\|X\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2 = n$$

$$X^T X = I$$

$$\langle v, w \rangle = \langle X^T v, w \rangle = \langle X v, X w \rangle$$

$$X^T v = v \quad \langle X X^T v \rangle$$

