

**[2(b)]**

Let  $F_1 = p^{-1}(p(C))$  and  $F_2 = p(C)$ , where  $C = \{z \mid p'(z) = 0\}$  is the critical set. Then  $p: \mathbb{C} \setminus F_1 \rightarrow \mathbb{C} \setminus F_2$  is a local homeomorphism (by the IFT), and for each  $y \in \mathbb{C} \setminus F_2$ ,  $\# p^{-1}(y) = 3$ . Thus  $p: \mathbb{C} \setminus F_1 \rightarrow \mathbb{C} \setminus F_2$  is a 3-fold covering.

**[3]**

Let  $H: \mathbb{I} \times \mathbb{I} \rightarrow X$  (cont.) be a free homotopy from  $a$  to  $b$  in  $X$ :

$$H(0, t) = H(1, t) \quad \forall t \in \mathbb{I}, \quad H(s, 0) = a(s), \quad H(s, 1) = b(s).$$

Let  $\tilde{a}$  be a closed lift of  $a$  ( $\phi \circ \tilde{a} = a$ ,  $\tilde{a}(0) = \tilde{a}(1) \stackrel{\text{def}}{=} \tilde{x}_0$ )

Then  $\tilde{x}_0 \in \phi^{-1}(x_0)$ ,  $x_0 = a(0) = a(1)$ .

Let  $\tilde{H}: \mathbb{I} \times \mathbb{I} \rightarrow \tilde{X}$  be the unique lift of  $H$  s.t.  $\tilde{H}(0, 0) = \tilde{x}_0$ .

Then  $s \mapsto \tilde{H}(s, 0)$  is a lift of  $a$  from  $\tilde{x}_0$ , hence equals  $\tilde{a}$  and:

$$\tilde{H}(1, 0) = \tilde{a}(1) = \tilde{a}(0) = \tilde{H}(0, 0).$$

want  $\tilde{H}(0, t) = \tilde{H}(1, t) \quad \forall t \in [0, 1]$ . Let  $\mathcal{G} \subset [0, 1]$  be the set of  $t$  where this holds (hence  $0 \in \mathcal{G}$ ). Let  $t_0 = \sup \mathcal{G}$ . More precisely:

$\mathcal{G} = \{t \in [0, 1] \mid \tilde{H}(0, \tau) = \tilde{H}(1, \tau) \quad \forall 0 \leq \tau \leq t\}$ . Then  $0 \in \mathcal{G}$  and  $\sup \mathcal{G} = t_0 \in \mathcal{G}$  (if  $t_n \nearrow t_0$ ,  $t_n \in \mathcal{G}$ , then  $t_0 \in \mathcal{G}$  by continuity of  $\tilde{H}$ .)

We claim  $t_0 = 1$ . If  $t_0 < 1$ ; let  $V \subset \tilde{X}$  be a nbhd of  $\tilde{a}_{t_0}(0) = \tilde{a}_{t_0}(1) \in \tilde{X}$  s.t.  $\phi|V$  is a homeo  $\tilde{V} \rightarrow \phi(V)$ .

(here  $a_{t_0}(s) = \tilde{H}(s, t_0)$ ) Then  $\exists \delta > 0$ , s.t.  $\tilde{H}(A_0 \sqcup A_1) \subset V$ ,  $(t_0 + \delta < 1)$

where  $A_0 = [0, \delta] \times (t_0 - \delta, t_0 + \delta)$  and  $A_1 = (1 - \delta, 1] \times (t_0 - \delta, t_0 + \delta) \subset \mathbb{I}$ .

Then for  $(s, t) \in A_0 \sqcup A_1$ :  $\tilde{H}(s, t) = (\phi|V)^{-1} H(s, t)$  (uniqueness of lift)

In part.  $\tilde{H}(0, t) = (\phi|V)^{-1} (H(0, t)) = (\phi|V)^{-1} (H(1, t)) = \tilde{H}(1, t)$  if  $|t - t_0| < \delta$ .

Thus  $[t_0, t_0 + \delta) \subset \mathcal{G}$ , contradicting  $t_0 = \sup \mathcal{G}$ . So  $t_0 = 1$ .

(2)

6.  $X$ : arb. topological space,  $f: X \rightarrow S^1$  cont, homotopic to a const.

Claim  $f$  admits a continuous lift  $\tilde{f}: X \rightarrow \mathbb{R}$ . (I.e.  $p \circ \tilde{f} = f$ ).

(If  $X$  sat. the hypotheses of the lifting theorem, this is easy.)

Pf Let  $H: X \times I \rightarrow S^1$  (cont.) be the homotopy:

$$H(x, 0) = y_0 \in S^1, \quad H(x, t) = f_t(x), \quad H(x, 1) = f_1(x) = f(x)$$

For each fixed  $x \in X$ , consider the curve  $a_x: I \rightarrow S^1$  from  $y_0$  to  $f(x)$ :

$a_x(t) = H(x, t) = f_t(x)$ . Let  $b(s) = e^{is}: \mathbb{R} \rightarrow S^1$  be the exponential covering. Note that  $p$  is a local isometry, and an isometry from intervals in  $\mathbb{R}$  of length  $< \pi$  to arcs in  $S^1$  of length  $< \pi$ . Say ~~if~~

$p(0) = y_0$ . Now for each  $x \in X$ , lift  $a_x$  to  $\tilde{a}_x: I \rightarrow \mathbb{R}$ ,  $\tilde{a}_x(0) = 0 \in \mathbb{R}$

Define  $\tilde{f}(x) = \tilde{a}_x(1) \in \mathbb{R}$ . Then  $p \circ \tilde{a}_x(t) = a_x(t) \rightsquigarrow = f_t(x)$ , so

$$\text{for } t=1 \quad p \circ \tilde{f}(x) = p \circ \tilde{a}_x(1) = a_x(1) = f_1(x) = f(x).$$

Continuity of  $\tilde{f}$

(Note: if  $A \subset \mathbb{R}$  is open, from  $p \circ \tilde{f} = f$  follows only  $\tilde{f}^{-1}(A) \subset f^{-1}(p(A))$   $\swarrow$  open, not enough for continuity in general).

Let  $\bar{x} \in X$ . By uniform continuity of the homotopy, we may find  $V_{\bar{x}} \subset X$  open

Let  $\epsilon > 0$ .

s.t.  $x \in V_{\bar{x}} \Rightarrow d_{S^1}(f_t(x), f_t(\bar{x})) < \epsilon \pi / 2, \quad \forall t \in [0, 1]$ .

Thus  $x \in V_{\bar{x}} \Rightarrow d_{S^1}(p(a_x(t)), a_{\bar{x}}(t)) < \epsilon \pi / 2 \quad \forall t \in [0, 1]$ .

By the isometry property of  $p$ , since  $\tilde{a}_x(0) = \tilde{a}_{\bar{x}}(0) = 0 \in \mathbb{R}$ , we have

for the lifts:  $|\tilde{a}_x(t) - \tilde{a}_{\bar{x}}(t)| < \epsilon \quad \forall t \in [0, 1]$ .

In particular, since  $\tilde{a}_x(1) = \tilde{f}(x)$  and  $\tilde{a}_{\bar{x}}(1) = \tilde{f}(\bar{x})$ , we have:

$$x \in V_{\bar{x}} \Rightarrow |\tilde{f}(x) - \tilde{f}(\bar{x})| < \epsilon,$$

establishing continuity of  $\tilde{f}$ .

[8] Want:  $f: M_g \rightarrow S^1$  not homotopic to a constant.

It's enough to have  $f_*: \pi_1(M_g) \rightarrow \pi_1(S^1) \approx \mathbb{Z}$  not trivial.  
(For  $g=1$ , this is easy).

Recall the wedge (bouquet) of  $g$  circles is a retract of  $M_g$  (see [Munkres, p. 375]  
for  $g=2$ )

$\pi_1(V_g S^1) \approx F_g$  (free group on  $g$  generators)

Let  $g: V_g S^1 \rightarrow S^1$  (cont.) send all but one of the circles to the basepoint,  
and be the identity on the remaining circle.

Then  $g_*: F_g \rightarrow \mathbb{Z}$  is non-trivial (indeed surjective).

Now consider  $M_g \xrightarrow{\tau} V_g S^1 \xrightarrow{g} S^1$ , where  $\tau$  is a retraction

Let  $f = g \circ \tau: M_g \rightarrow S^1$ . Then  $f_* = g_* \circ \tau_*$ :

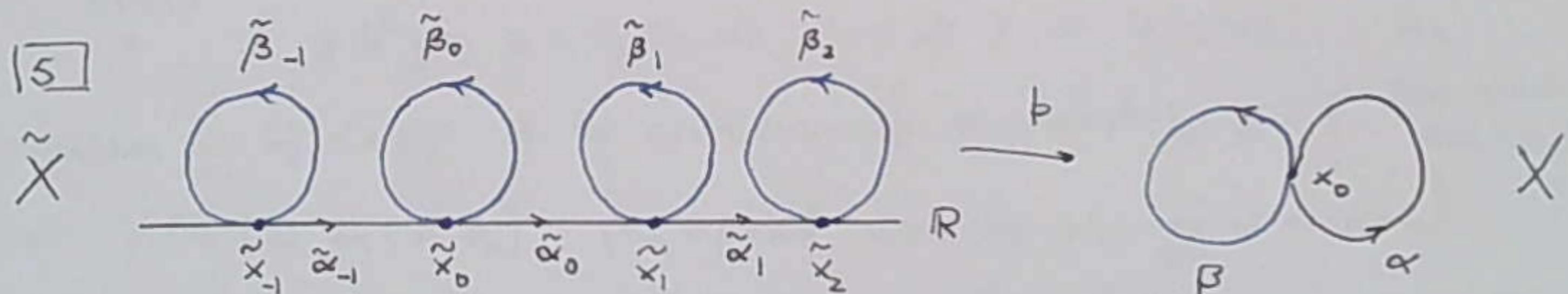
$$\pi_1(M_g) \xrightarrow{\tau_*} F_g \xrightarrow{g_*} \mathbb{Z}$$

Recall  $j_*: \pi_1(V_g S^1) \rightarrow \pi_1(M_g)$  is injective; indeed  $r_* \circ j_*$  is the  
identity on  $F_g = \pi_1(V_g S^1)$ .

Thus  $f_* = g_* \circ \tau_*: \pi_1(M_g) \rightarrow \mathbb{Z}$  cannot be trivial; if it were, then also

$f_* \circ j_*: F_g \rightarrow \mathbb{Z}$  would be trivial. But:

$$f_* \circ j_* = g_* \circ (\tau_* \circ j_*) = g_*: F_g \rightarrow \mathbb{Z}, \text{ which is not trivial.}$$



Let  $b: \tilde{X} \rightarrow X$  be the standard cover of the right (black) circle by  $\mathbb{R}$ ,  
and map each blue circle tangent to  $\mathbb{R}$  at  $\tilde{x}_n = n \in \mathbb{Z}$  homeomorphically to the  
left (blue) circle on  $X$ .  $b^{-1}(x_0) = \{\tilde{x}_n; n \in \mathbb{Z}\}$  with  $\tilde{x}_n = n$ .

The lift of  $\alpha$  from  $\tilde{x}_n$  is  $\tilde{\alpha}_n$  (all open), while the lift of  $\beta$  from  $\tilde{x}_n$  is  
 $\tilde{\beta}_n$  (all closed); thus  $b$  is regular.

Another reason is that translations of  $\tilde{X}$  (by an integer) along  $\mathbb{R}$  are  
covering automorphisms, so  $\text{Aut}(\tilde{X}|X)$  acts transitively on the fibers.

Let  $\begin{cases} \gamma_n = \tilde{\alpha}_0 * \dots * \tilde{\alpha}_n & \text{if } n \geq 0 \\ \gamma_n = \bar{\tilde{\alpha}}_{-1} * \bar{\tilde{\alpha}}_{-2} * \dots * \bar{\tilde{\alpha}}_n & \text{if } n < 0 \end{cases}$  (( $n+1$ )-fold lift of  $\alpha$  from  $\tilde{x}_0$ )  
 $\quad \quad \quad$  ( $n$ -fold lift of  $\bar{\alpha}$  from  $\tilde{x}_0$ )

Then  $\pi_1(\tilde{X}, \tilde{x}_0)$  is freely generated by the homotopy classes in  $\tilde{X}$ :  
 $\quad \quad \quad$  (of loops based at  $\tilde{x}_0$ )

$$\left\{ [\gamma_n * (\tilde{\beta}_{n+1} * \bar{\gamma}_n)]_{\tilde{X}} ; n \in \mathbb{Z} \right\} \cup \left\{ [\tilde{\beta}_0]_{\tilde{X}} \right\} \cup \left\{ [\gamma_n * (\beta_n + \bar{\gamma}_n)]_{\tilde{X}} ; n < 0 \right\}$$

~~not 0~~  
n ≥ 0

$$\pi_1(X, x_0) = F_2 \langle a, b \rangle \quad \text{where } a = [\alpha]_X, b = [\beta]_X. \quad (\text{free group})$$

action of  $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$

$$p_* [\gamma_0 * (\tilde{\beta}_1 * \bar{\gamma}_0)]_{\tilde{X}} = ab a^{-1}$$

$$p_* [\gamma_1 * (\tilde{\beta}_2 * \bar{\gamma}_1)]_{\tilde{X}} = a^2 b a^{-2}$$

$$p_* [\gamma_n * (\tilde{\beta}_{n+1} * \bar{\gamma}_n)]_{\tilde{X}} = a^n b a^{-n} \quad n \geq 0$$

$$p_* [\tilde{\beta}_0]_{\tilde{X}} = b$$

$$p_* [\gamma_n * (\beta_n + \bar{\gamma}_n)]_{\tilde{X}} = a^n b a^{-n}, \quad n < 0$$

Thus  $H(\tilde{x}_0) = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$  is given by generators

$$H(\tilde{x}_0) = \langle b, ab a^{-1}, a^2 b a^{-2}, \dots, a^{-1} b a, a^{-2} b a^2, \dots \rangle$$

CLAIM  $= \langle g b^n g^{-1} ; g \in F_2 \langle a, b \rangle, n \in \mathbb{Z} \rangle = N(\langle b \rangle)$ , the  
 normalizer in  $F_2 \langle a, b \rangle$  of the cyclic subgroup generated by  $b$ . (The claim is easily checked)

Thus  $H(\tilde{x}_0) \trianglelefteq \pi_1(X, x_0)$  (as expected, since the covering is normal)

$$\underline{\text{Claim}} \quad \text{Aut}(\tilde{X} | X) = \pi_1(X) / H(\tilde{x}_0) = \frac{F_2 \langle a, b \rangle}{N(\langle b \rangle)} \approx \langle a \rangle (\cong \mathbb{Z})$$

↑ claim (translations!)

To see this, consider the homomorphism  $\phi : F_2 \langle a, b \rangle \rightarrow \langle a \rangle$  defined by  $\begin{cases} \phi(a) = a \\ \phi(b) = e \end{cases}$   
 (clearly onto)  $\langle b \rangle \subset \text{Ker } \phi$ , so  $N(\langle b \rangle) \subset \text{Ker } \phi$ .

In fact if  $\phi(g) = e$ ,  $\phi \in N(\langle b \rangle)$  Let  $g = a^{n_1} b^{m_1} \dots a^{n_p} b^{m_p}$  (say).  $n_i, m_i \in \mathbb{Z} - \{0\}$

Then  $\phi(g) = e$  iff  $n_1 + \dots + n_p = 0$

$$\phi=2: n_1 + n_2 = 0 \quad g = a^{n_1} b^{m_1} a^{-n_1} \in N\langle b \rangle$$

$$\phi=3: n_1 + n_2 + n_3 = 0 \quad g = a^{n_1} b^{m_1} a^{n_2} b^{m_2} a^{n_3} b^{m_3} = a^{n_1} b^{m_1} a^{n_2} b^{m_2} a^{-n_2} a^{-n_1} b^{m_3}$$

$$= (a^{n_1} b^{m_1} a^{-n_1})(a^{n_1+n_2} b^{m_2} a^{-(n_1+n_2)}) b^{m_3} \in N\langle b \rangle$$

etc.