Finite Complement Topology

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- Is the finite Complement topology on $\mathbb{R}$ Hausdorff?

  No, It is not Hausdorff.
  Let τ represent the finite complement topology on $\mathbb{R}$. U is open in τ if $U = \emptyset$ or $\mathbb{R} - U$ is finite. Hence, we can write every non-empty open set in τ as $\mathbb{R} - F$ where F is finite. In order to show that $(\mathbb{R}, \tau)$ is non-Hausdorff, we would prove that the intersection of two arbitrary open sets in τ is non-empty. This would then imply that no two points have disjoint neighborhoods.

  Proof. Let U_1 and U_2 be two open sets in τ. U_1 and U_2 can be written as $\mathbb{R} - F_1$ and $\mathbb{R} - F_2$ where F_1 and F_2 are finite sets.

  $$U \cap V = (\mathbb{R} - F_1) \cap (\mathbb{R} - F_2) = \mathbb{R} - (F_1 \cup F_2)$$

  Since, $F_1 \cup F_2$ is finite, $\mathbb{R} - (F_1 \cup F_2)$ is non empty □

- Is this topology finer or coarser than the usual one?

  Coarser.

  Proof. Let τ_f, τ represent the finite complement topology and standard topology on $\mathbb{R}$ respectively. If $U \in \tau_f$, $U \in \tau$ since $\mathbb{R} - U$ is a finite set which is closed according to the usual topology. On the other hand, consider $(a, b) \in \tau$ where $a, b \in \mathbb{R}$ and $a < b$. $\mathbb{R} - (a, b)$ isn't finite, hence $(a, b) \notin \tau_f$. Since, τ_f has fewer open sets than τ, it is coarser □

- What does $\lim x_n = a$ mean in this topology?

  It means that x_n eventually becomes constant and take on the value a infinitely many times. To see this, let $U \in \tau_f$ be an open set around a. By definition , $\mathbb{R} - U$ is finite. It follows then that $x_n \to a$ since only finitely many elements of x_n are in $\mathbb{R} - U$. Consequently, for $b \neq a$, $V = \mathbb{R} - \{a\}$ is an open set in τ_f such that $b \in V$ and V misses infinitely many terms of x_n. This means that x_n cannot converge to b and in particular, no subsequence of x_n converges to b.