Analysis Homework 1

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Problem 12

Lemma. *If* (X, d) *is a metric space, then* $(X, min\{d, 1\})$ *is also a metric space.*

Proof. Let $\rho(x, y) = min\{d(x, y), 1\}$. We note that the first two conditions of a metric are trivially true. Thus, we only need to show that the triangle inequality holds for $min\{d, 1\}$ Let $x, y, z \in X$. We need to show

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y)$$

We know

$$d(x, y) \leq d(x, z) + d(z, y)$$

Now, assume $d(x, y) \geq 1$, thus $\rho(x, y) = 1$ and one of the following must hold.

Case 1: let $d(x, z) < 1$ and $d(z, y) < 1$. Thus, $\rho(x, z) = d(x, z)$ and $\rho(z, y) = d(z, y)$, which implies that $1 = \rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Case 2: Without loss of generality, let $d(x, z) \geq 1$ and $d(z, y) < 1$. Thus $\rho(x, z) = 1$ and $\rho(z, y) = d(z, y)$, which implies that $1 = \rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Case 3: let let $d(x, z) \geq 1$ and $d(z, y) \geq 1$. Thus, $\rho(x, z) = 1$ and $\rho(z, y) = 1$, which implies that $1 = \rho(x, y) \leq \rho(x, z) + \rho(z, y) = 2$.

Now, assume that $d(x, y) < 1$, thus $\rho(x, y) = d(x, y)$. If either $d(x, z) \geq 1$ or $d(z, y) \geq 1$ then the triangle inequality is trivial. So assume $d(x, z) < 1$ and $d(z, y) < 1$. Thus, $\rho(x, z) = d(x, z)$ and $\rho(z, y) = d(z, y)$. Since d is a metric, the triangle inequality holds here as well.

Therefore, the triangle inequality holds in all possible cases, meaning If (X, d) is a metric space, then $(X, min\{d, 1\})$ is also a metric space. $\square$

Theorem. *(X, d) and (X, ρ) where $\rho = min\{d, 1\}$ are equivalent metric spaces*

Proof. Let U_1 denote the set of all d-balls $B_d(x)$, centered at x where $x \in X$ with radius $d > 0$ in (X, d). Thus we can form a basis for a topology on (X, d) using the elements of U_1. Now, we let $U_2 \subset U_1$ such that U_2 is the set of all d-epsilon balls in (X, d) such that $0 < d \leq 1$. Now, we note that we can produce U_1 by taking the union of elements in U_1. Thus, the basis formed by U_2 generates the same topology as U_1. Now, note that U_2 is just the set of all ρ-balls $B_\rho(x)$. Thus the elements of U_2 form a basis for a topology on both (X, d) and $(X, min\{d, 1\})$.

Therefore, since the metrics on (X, d) and $(X, min\{d, 1\})$ generate the same topology, they are equivalent metric spaces. □