MS61 – FINAL EXAM – solutions

$\boxed{1}$ (i) Let $K \subset \mathbb{R}$ be compact, $\varepsilon > 0$. Say $|x-c| \le R$ for $x \in K$.

$R = R(K)$

Then
$$|f_n(x) - L| \le |f_n(x) - f_n(c)| + |f_n(c) - L|$$
$$\le \int_x^c |f_n'(t)|\,dt + |f_n(c) - L|$$
$$\le \underbrace{\Big(\sup_{x\in K} |x - c|\Big)}_{\le R} \Big(\sup_{t \in K} |f_n'(t)|\Big) + |f_n(c) - L|$$

choose $N = N(K)$ so that $|f_n(c) - L| < \frac{\varepsilon}{2}$ and $|f_n'(t)| < \frac{\varepsilon}{2R}$ for $n \ge N$ and $t \in K$.

Then $|f_n(x) - L| < \varepsilon$ for $x \in K$.

(ii) Let p be a polynomial so that $|p(t) - f(t)| \le \varepsilon$ for $t \in [0,1]$. By linearity $\int_0^1 p(t) f(t)\,dt = 0$. But $\int_0^1 f^2(t)\,dt = \int_0^1 f(t)\,(f(t) - p(t))\,dt$ $\le \varepsilon \int_0^1 |f(t)|\,dt$. Since ε is arbitrary, $\int_0^1 f^2(t)\,dt = 0$, so $f \equiv 0$ in $[0,1]$.

$\boxed{2}$. Let $f \in C(X,Y)$, $V \subset C(X,Y)$ an open nbd of f (which we may assume is a basic nbd: $V = S(C_1, V_1) \cap \dots \cap S(C_n, V_n)$ $\left\{ \begin{array}{l} C_i \text{ compact in } X \\ V_i \subset Y \text{ open} \\ f(C_i) \subset V_i \end{array} \right.$

<u>want</u> $U \subset C(X,Y)$ nbd of f s.t. $\overline{U} \subset V$

We may find $U_i \subset Y$ s.t. $f(C_i) \subset U_i \subset \overline{U_i} \subset V_i$ (since Y regular)

Let $U = S(C_1, U_1) \cap \dots S(C_n, U_n) \subset C(X,Y)$ (open nbd. of f)

$\mathcal{F} = \{ g \in C(X,Y) ;\ g(C_1) \subset \overline{U}_1, \dots, g(C_n) \subset \overline{U}_n \}$ $U \subset \mathcal{F}$.

<u>Claim</u> (i) $\mathcal{F} \subset V$ (since $\overline{U}_i \subset V_i$) (ii) $\mathcal{F}$ is closed in $C(X,Y)$.

Let $g \in C(X,Y) \setminus \mathcal{F}$. Then $g(C_{i_0}) \not\subset \overline{U}_{i_0}$ for some i_0: $\exists\, x_0 \in C_{i_0}$, $g(x_0) \notin \overline{U}_{i_0}$

Let $W \subset Y$ be a nbd of $g(x_0)$ disj. from $\overline{U}_{i_0}$. Then $S(x_0, W)$ is a nbd of g, and if $h \in S(x_0, W)$, $h \notin \mathcal{F}$. Hence the complement of $\mathcal{F}$ is open.

Thus $\overline{U} \subset \overline{\mathcal{F}} = \mathcal{F} \subset V$.

$\boxed{4}$ (i) Take the metric $d = \max\{d_X, d_Y\}$ on $X \times Y$. Suppose $\exists$ a ball $B(x_0, y_0) = B^X(x_0) \times B^Y(y_0)$ with no other points of $X \times Y$. Then either $B^X(x_0)$ has no other pts of X or $B_Y(y_0)$ has no other points of Y; contradiction in both cases.

$\boxed{3}$ (i) If $P = X \times Y$ is connected, so are X and Y (images of P under cont. maps). For the converse: $\{x \times Y\}$ homeo to Y, $X \times \{y_0\}$ homeo to X — hence both connected, with the point (x, y_0) in common; so their union T_x is connected. Now $P = \bigcup_{x \in X} T_x$, and all the T_x have the common point (x_1, y_0). Hence P connected.

(ii) Let $U \subset X$ be open, connected. Fix $x_0 \in U$ and let $C = \{x \in U;\ x$ is connected to x_0 by a path in $U\}$. Then C is open in U (since X is loc. path connected) and also closed in U: if $x_n \to x_0 \in U$, $x_n \in C$ then let $B_{x_0}(\varepsilon) \subset U$ be path connected. $x_n \in B_{x_0}(\varepsilon)$ for $n \geq N$, so connect x_n to x_0 by a path in $B_{x_0}(\varepsilon)$ to see $x_0 \in C$. Thus $C = U$, since U is connected.

$\boxed{5}$ (i) Let $\mathcal{C}_0 = \{(n - \frac{1}{2}, n + \frac{1}{2});\ n \in \mathbb{Z}\}$, $\mathcal{C}_1 = \{(n, n+1);\ n \in \mathbb{Z}\}$. Then $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$ is a cover of $\mathbb{R}$ by open intervals. The sets in $\mathcal{C}_0$ are all disjoint from each other, and same for $\mathcal{C}_1$. Thus each $x \in \mathbb{R}$ is in at most two sets of $\mathcal{C}$. Each set in $\mathcal{C}$ has diameter 1. Thus letting $\mathcal{C}(\delta) = \{\delta I;\ I \in \mathcal{C}\}$, $\mathcal{C}(\delta)$ is an open cover of $\mathbb{R}$ of order 2, by sets of diameter δ.

(ii) Let $X \subset \mathbb{R}$ compact, $\mathcal{C}$ an open cover of X, 2δ a Lebesgue number for $\mathcal{C}$. Thus any open cover of X by sets w/ diameter $\leq 2\delta$ refines $\mathcal{C}$. Let $\mathcal{B}(\delta)$ be the cover of X obtained by intersecting the sets of $\mathcal{C}(\delta)$ (part (i)) w/ X. Then $\mathcal{B}(\delta)$ refines $\mathcal{C}$, and has order ≤ 2.

$\boxed{6}$ It suffices to show p is injective (since it is cont, open and surjective). By contradiction, assume we have $\tilde{x}_0 \neq \tilde{x}_1 \in X$ s.t. $p(\tilde{x}_0) = p(\tilde{x}_1) = x_0 \in Y$. Let $\tilde{\gamma}$ be a path in X from $\tilde{x}_0$ to $\tilde{x}_1$. Then $\gamma = p \circ \tilde{\gamma}$ is a loop in X based at x_0, hence homotopic (w/ basepoint) to the constant loop e_{x_0}. The unique lift of e_{x_0} to X is $e_{\tilde{x}_0}$, the constant loop at $\tilde{x}_0$. Yet the homotopy also lifts to X, w/ fixed endpoints. Hence $\tilde{x}_0 = \tilde{x}_1$, contradiction.

$\boxed{4}$ (ii) Suppose the conn. component of (x_0, y_0) contains (x_1, y_1):

$$(x_1, y_1) \in \bigcup \{C;\ C \subset X \times Y \text{ connected};\ (x_0, y_0) \in C\}.$$

Thus $\exists\, C \subset X \times Y$ connected, w/ (x_1, y_1) & $(x_0, y_0) \in C$. Then $C_X = \pi^X(C)$ and $C_Y = \pi^Y(C)$ are connected subsets of X (resp. Y), ~~containing~~ with $x_0, x_1 \in X$ resp $y_0, y_1 \in Y$. So $x_0 = x_1$ and $y_0 = y_1$.

(iii) Any compact, perfect, totally disconnected metric sp. (nonempty) is homeomorphic to C. And by (i) & (ii) $C \times C$ (is compact and) has those properties.