Connectivity of GL_n^+ (~~positive~~ invertible $n\times n$ real matrices with $\det > 0$)

$\boxed{\text{Step 1}}$ Let $\lambda \in C(\mathbb{I}, S^N)$ be a path on the N-sphere $S^N \subset \mathbb{R}^{N+1}$.

~~Recall~~ Let $\{e_1, \dots, e_N\}$ be an o.n. basis of $T_{\lambda(0)} S^N$, the tangent space. Then $\exists\ v_i : \mathbb{I} \to \mathbb{R}^{N+1}$ (cont.) $i = 1, \dots, N$ so that $v_i(0) = e_i$ and $\{v_1(t), \dots, v_N(t)\}$ is an o.n. basis of $T_{\lambda(t)} S^N$.

Proof (i) Assume $\lambda(0) = e^0_{N+1}$ (std basis vector in $\mathbb{R}^{N+1}$),

so $T_{\lambda(0)} S^N \simeq \mathbb{R}^N \cong \{v \in \mathbb{R}^{N+1},\ \langle v, e^0_{N+1}\rangle = 0\}$. If $q \in S^N$ and $\langle q, e^0_{N+1}\rangle > 0$, the orthogonal projection from $\mathbb{R}^N$ to $T_q S^N$:

$$P_q(v) = v - \langle v, q\rangle q$$

is an isomorphism. (check: it's easy.). So it takes the std basis vectors $\{e_1^0, \dots, e_N^0\}$ of $\mathbb{R}^N$ to a basis of $T_q S^N$. Recall the Gram-Schmidt process defines a map

$$GS : GL_n^+ \longrightarrow SO(n)$$

$\{w_1, \dots, w_n\}$ basis $\qquad [w_1 | \cdots | w_n] \longmapsto [e_1 | \cdots | e_n] \qquad \{e_1, \dots, e_n\}$ o.n.

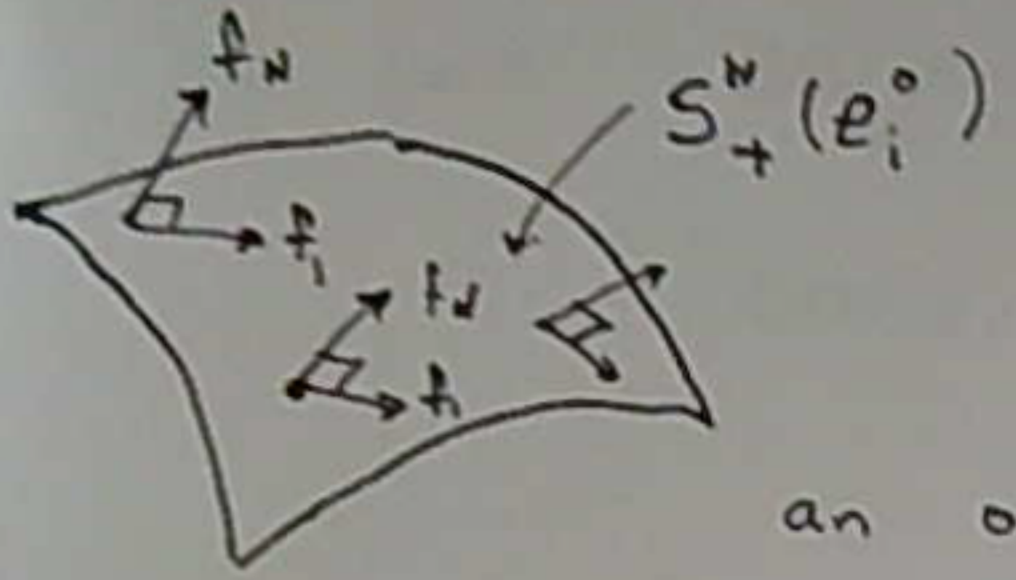
(we identify bases w/ matrices via column vectors).

This map is continuous (for the topologies induced from $\mathbb{R}^{n^2}$) and the identity on $SO(n)$ (doesn't change bases that are already orthonormal), so is a retraction.

Thus for $q \in S^N_+(e^0_{N+1}) = \{q \in S^N \mid \langle q, e^0_{N+1}\rangle > 0\}$ (the 'open upper hemisphere' defined by e^0_{N+1}), we have the o.n. basis for $T_q S^N$:

$$f_i(q) = (GS \circ P_q)(e_i^0) \quad i = 1, \dots, N,$$

this is a 'moving positive frame' for $T_q S^N$, depending continuously (in fact, smoothly) on q. So we may cover S^N by $2(N+1)$ 'cells' (open hemispheres, homeo to the N-ball in $\mathbb{R}^N$) $S^N_+(e_i^0)$, $S^N_-(e_i^0)$ $(i = 1, \dots, N+1)$, so that each of them carries a field of positive o.n. frames for the tangent space, depending cts'ly on the basepoint.



So given this cover $\{U_i\}_{i=1}^{2(N+1)}$ of S^N and a path $\lambda : I \to S^N$, when $\lambda(t) \in U_i$ we have an o.n. basis $\mathcal{B}_i(t)$ of $T_{\lambda(t)} S^N$. Divide I into open intervals I_j, so that $\lambda(t)$ is in one of the open hemispheres when $t \in I_j$, and so that each $t \in I$ is in at most two I_j. On $I_1 \cap I_2$, we have two o.n. bases $\mathcal{B}_1(t)$, $\mathcal{B}_2(t)$ for $T_{\lambda(t)} S^N$. So $\exists\, A(t) \in SO(N)$ so that $\mathcal{B}_2(t) = A(t)\,\mathcal{B}_1(t)$ (column matrices). Replace (for $t \in I_2 \cap I_1$) $\mathcal{B}_2(t)$ by $\tilde{\mathcal{B}}_2(t) = A^{-1}(t)\,\mathcal{B}_2(t)$, to extend $\mathcal{B}_1(t)$ continuously to I_2. Continue in this fashion: $\mathcal{B}_1(t)$ is extended to o.n. frames $\mathcal{B}(t)$, $t \in I$. □

Step 2 Let $\{u_1, \dots, u_m\}$, $\{v_1, \dots, v_m\}$ be positive o.n. bases of $\mathbb{R}^m$. Then $\exists$ cont. paths $w_i(t)$ (of positive o.n. bases) joining one to the other. (Equivalently, via column matrices: $SO(m)$ is path connected.)

Proof (induction on m). Assume valid for $m-1$. Let $\lambda_m : [0,1] \to S^{m-1}$ join $u_m \in S^{m-1}$ to $v_m \in S^{m-1}$. From step 1, we have $\{w_i(t)\}_{i=1}^{m-1}$, o.n. bases ~~paths~~ of $T_{\lambda_m(t)} S^{m-1}$ starting at $\{u_1, \dots, u_{m-1}\}$. The matrix from $\mathcal{B}_{m-1}(1) = \{w_1(1), \dots, w_{m-1}(1)\}$ to $\{v_1, \dots, v_{m-1}\}$ is $A \in SO(m-1)$, and then

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} \in SO(m) \text{ is the matrix from } \{w_i(1), \lambda(1) = v_m\} \text{ to } \{v_i, \overset{v_m}{\overset{\shortparallel}{\lambda(1)}}\}$$

Thus (by induction) $\exists\, A : [1,2] \to SO(m-1)$ w/ $A(1) = I$, $A(2) = \tilde{A}$. Then letting $\mathcal{B}(t) = A(t)\,\mathcal{B}_{m-1}(t)$ for $t = [1,2]$, we get $w_1(2) = v_1, \dots, w_m(2) = v_m$.

Remark (Advanced perspective)

The map $p : SO(N+1) \longrightarrow S^N$, $p(A) = A\, e^0_{N+1}$ is the identity on the subgroup $SO(N)$ of $SO(N+1)$.

In fact $SO(N+1)$ has the structure of a 'locally trivial fibration':

$$
\begin{tikzcd}
SO(N) \arrow[r, hook] & SO(N+1) \arrow[d, "p"] \\
& S^N
\end{tikzcd}
$$

with base S^N and typical fiber $SO(N)$. (Indeed a "principal fiber bundle", since $SO(N)$ is a Lie group.). These fibrations have the "path lifting property" (not unique!). Indeed step 1 is essentially a proof of path lifting (and part (i) is the 'locally trivial' statement), while step 2 is the fact that a fibration with (path) connected base and connected fiber has connected total space.

Step 3 $\{a_1, \dots, a_m\}$, $\{b_1, \dots, b_m\}$ positive bases of $\mathbb{R}^m$.

$\Rightarrow \exists\, c_1, \dots, c_m : [0,2] \to \mathbb{R}^m$ cont. from a_i to b_i, each $c_i(t)$ a positive basis of $\mathbb{R}^m$ (i.e. $GL_+(m)$ is path connected).

Proof ETS $\exists\, a_i(t)$ from a_i to a pos. orthonormal basis $\{e_i\}$ (proof by induction on m). Let $E = \operatorname{span}\{a_1, \dots, a_{m-1}\} \subset \mathbb{R}^m$, e a unit normal vector.

Writing $e = \alpha_1 a_1 + \dots + \alpha_m a_m$ we ~~may~~ assume $\langle a_m, e\rangle > 0$ (or replace e by $-e$)

Then let $a_m(t) = (1-t)a_m + te$, $t \in [0,1]$. We have:

$$\langle a_m(t), e\rangle = (1-t)\langle a_m, e\rangle + t > 0 \quad \forall t, \text{ so}$$

$\{a_1, \dots, a_{m-1}, a_m(t)\}$ is a basis of $\mathbb{R}^m$, ending with $\{a_1, \dots, a_{m-1}, e\}$.

Now (by induction) join $\{a_1, \dots, a_{m-1}\}$ to $\{e_1, \dots, e_{m-1}\}$ o.n. basis for E. We get (combining the 2 homotopies) $c_1, \dots, c_m : [0,2] \to \mathbb{R}^m$ cont., ending in an o.n. basis.

Remark Note that if $\{a_1, \dots, a_m\}$ is already orthonormal, this path is constant. This exhibits $SO(m)$ as a deformation retract of $GL_+(m)$.