Complete metrizability

(of subsets of a complete metric space)

(Q) When is a topological space completely metrizable?

(i.e. its topology is given by a complete metric)

Thm 1 If $A \subset M$ (M, d) complete metric $\left(A = \overset{\text{example}}{\mathbb{R}^2 \setminus \{0\}}\right)$

A is completely metrizable $\Leftrightarrow$ A is a G_δ subset of M.

idea for $(\Leftarrow)$ suppose A is open in (M, d)

Consider $f(x) = d(x, M \setminus A)$ and $\varphi(x) = \dfrac{1}{f(x)}$

$\varphi \in C(A; \mathbb{R}_+)$

the graph of φ:

$G = \{(x,t) \in A \times \mathbb{R};\ t = \varphi(x)\}$ closed in $M \times \mathbb{R}$ (complete)

Hence G is complete metric (metric induced from $M \times \mathbb{R}$)

$p : G \longrightarrow A$ is a homeo.

$(x,t) \longmapsto x$

hence A is complete metrizable

Exercise 1: Complete the details of this outline of the proof.

(i) The graph G of φ is a closed subset of ~~$A\times\mathbb{R}$~~ $M\times\mathbb{R}$ with the metric:

$$\rho((x,t),(y,s)) = d(x,y) + |s-t| \quad x,y\in M \quad s,t\in\mathbb{R}$$

(Hint: $G = \{(x,t) \mid t\, f(x) = 1\}$)

(ii) The projection $M\times\mathbb{R} \to M$ restricted to G:

$$p : G \longrightarrow A \qquad (x,\varphi(x)) \longmapsto x$$

is a homeomorphism from G to A.

Note that the metric ρ is complete, so this shows A is completely metrizable. An explicit complete metric on A is given by:

$$d_1(x,y) = d(x,y) + \left| \frac{1}{d(x,A^c)} - \frac{1}{d(y,A^c)} \right|$$

$(A^c = M\setminus A)$.

Ex. 2 Metric in a countable product of metric spaces $M = \prod_{i=1}^{\infty} M_i$ (M_i, d_i)

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}$$

As we know, $\dfrac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}$ and $d_i(x_i,y_i)$ are equivalent metrics on M_i

Prove: (M,d) is complete $\Leftrightarrow$ each (M_i,d_i) is complete.

Hint ($\Leftarrow$) Let (x_n) Cauchy on M. Note (prove) $p_i : (M,d) \to (M_i, d_i)$ is unif. cont. Thus (x_n^i) is Cauchy in (M_i, d_i), for all $i\geq 1$.

($\Longrightarrow$) Embed (M_{i_0}, d_{i_0}) into (M,d) isometrically, in a natural way.

~~Ex 3~~ Now assume $A \subset M$ is a G_δ set; (M, d) complete.

$$A = \bigcap_{i \geq 1} U_i \qquad U_i \subset M \text{ open}.$$

Claim: A is homeomorphic to a complete metric space.

Pf: Find a metric d_i in U_i so that (U_i, d_i) is complete.

Then $U = \prod_{i=1}^{\infty} U_i$ is complete (for the metric d in Ex. 2).

Ex. 3: Let $\Delta = \{ x \in U \mid x_1 = x_2 = \cdots \}$ be the diagonal (recall all $x_i \in M$)

Show (*) $g : A \longrightarrow U$
$g(x) = (x, x, x, \dots)$ is an embedding, and $g(A) = \Delta$

Since Δ is a closed subset of a complete metric space, it is complete for the metric d. Hence g induces a complete metric on A.

Remark

The converse ($A \subset M$ completely metrizable $\Rightarrow$ A is a G_δ in M) is harder to prove.

Example (i) $\mathbb{Q} \subset \mathbb{R}$ is not completely metrizable(*) (since $\mathbb{Q}$ is not a G_δ)

(ii) $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ is completely metrizable (since $\mathbb{I}$ is a G_δ).

(*) Also follows from the fact $\mathbb{Q}$ is countable w/o isolated pts and problem 3.9

<u>Theorem</u> X metric, Y complete metric, $A \subset X$ arbitrary

$f : A \longrightarrow Y$ cont.

Let $G = \{ x \in \bar{A} \mid \lim_{a \to x} f(a) \text{ exists} \}$

Then

$\boxed{\text{Ex 4}}$ (i) f extends continuously to G (exercise 4)

(ii) G is a G_δ set in X

<u>Proof of (ii)</u>

Let $A_n = \{ x \in \bar{A} \mid \exists\, U_x \text{ nbd of } x \,;\ \operatorname{diam}(f(A \cap U_x)) < \frac{1}{n} \}$.

Then $G = \bigcap_{n \geq 1} A_n$.

Each A_n is open in $\bar{A}$: if $x \in A_n$, then any $y \in \bar{A} \cap U_x$ is also in A_n. Thus $A_n = \bar{A} \cap U_n$ for some U_n open in X.

So $G = \bigcap_{n \geq 1} A_n = \bar{A} \cap \bigcap_{n \geq 1} U_n$, and $\bar{A}$ is a G_δ.

<u>Corollary</u>

X complete, $A \subset X$ completely metrizable. Then A is a G_δ set.

<u>Proof</u> Apply the theorem to $id : (A, d) \longrightarrow (A, d_A)$, where d_A is a complete metric on A. We have a cont. extension to:

$$G = \{ x \in \bar{A} \mid \forall n \geq 1\ \exists\, U_x \text{ nbd of } x \text{ in } X;\ \operatorname{diam}_{d_A}(A \cap U_x) \leq \tfrac{1}{n} \}$$

G is a G_δ. Claim: $G = A$.

Let $x \in G$

Proof: We may find $r_n \downarrow 0_+$ so that

$$\operatorname{diam}_{d_A} \left(A \cap B_d(x, r_n) \right) \leq \frac{1}{n}.$$

and a sequence $x_n \in B_d(x, r_n) \cap A$ w/ $d_A(x_n, x_m) \leq \frac{1}{n}$ if $m \geq n$

(since $x \in \overline{A}$).

Thus $x_n \xrightarrow{d} x$; and (x_n) is d_A - Cauchy, so $x_n \xrightarrow{d_A} a$, for some $a \in A$.

Since d, d_A define the same Hausdorff topology on A,

$d\text{-}\lim x_n = a$, so $x = a$ and $x \in A$.