1 (i) $M = \bigcup_{n\geq 1} K_n$, K_n cpct metric $\Rightarrow \exists\, \mathcal{B}_n$ ctble basis of K_n.

Then $\mathcal{B} = \bigcup_{n\geq 1} \mathcal{B}_n$ is a ctble basis of M: if $U \subset M$ is open,

$U = \bigcup_{n\geq 1} U_n$ where $U_n = U \cap K_n$ is a union of sets in $\mathcal{B}_n$.

(ii) Let $\mathcal{B}_\omega$ be a ctble basis of nbds of ω in M^* (exists since M is σ-cpct).

Then $\mathcal{B} \cup \mathcal{B}_\omega$ is a ctble basis of open sets in M^*: $V \subset M^*$ open can be written as $V = (V \cap K_{n+1}) \cup (V \cap [(M \setminus K_n) \sqcup \omega])$, the first open in K_{n+1}, the second contained in an open nbd of ω, hence a union of sets in $\mathcal{B}_\omega$.

So M^* is 2^{nd} ctble and normal (given cpct.), hence metrizable [Urysohn].

2 Let U, V be disjoint open nbds. of x, y (resp.) (X Hausdorff).

Note $x \notin \overline{V}$, otherwise any nbd of x would intersect V.

X regular $\longrightarrow$ $\exists\, U_1$ nbd of x, $V_1 \supset \overline{V}$ s.t. $U_1 \cap V_1 = \emptyset$

We have $\overline{U_1} \cap \overline{V} = \emptyset$: if z is in the intersection, any nbd. of z intersects U_1. But $z \in V_1$ and $U_1 \cap V_1 = \emptyset$.

3 (i) Let $\mathcal{B}_n$ be a ctble. basis for X_n. Then $\mathcal{S} = \bigcup_{n\geq 1} \pi_n^{-1}(\mathcal{B}_n)$ is a ctble subbasis for $X = \prod_{n\geq 1} X_n$, $\pi_n : X \longrightarrow X_n$ the std projection.

Thus $\mathcal{B} = \{ U_1 \cap \cdots \cap U_N \,;\, N \geq 1,\ U_i \in \mathcal{S} \}$ is a ctble basis for X.

(ii) Let $P = \prod_{f \in \mathcal{F}} I_f$, $I_f = [\inf f, \sup f] \subset \mathbb{R}$. P is a ~~ctble~~ product of ctbly many intervals, hence 2^{nd} ctble by (i). The embedding $e : X \to P$ is given by $[e(x)]_f = f(x) \in I_f$, and $\hat{X} = \overline{e(X)}$ (closure in P) is a compact'n of X.

(iii) $\hat{X}$ is 2^{nd} countable (subspace of P, 2^{nd} ctble.) and compact (hence normal).

Thus $\hat{X}$ is metrizable (Urysohn).

4. (i) Let $U \subset X$ open, $V = f(U^c)^c \subset Y$. $f(U^c)$ is closed, so V is open.
If $y \in V$, $y \notin f(U^c)$, so $y \in f(U)$ (since f is surjective). So $V \subset f(U)$.

<u>Lemma</u> X Hausdorff, $A, B \subset X$ compact disjoint
$\Longrightarrow \exists\ U \supset A,\ V \supset B$ open disjoint

<u>Proof</u> Fix $b \in B$. Find, for each $a \in A$ nbds U_a of a, V_a of b, disjoint.
By compactness, $A \subset \bigcup_{i=1}^{N} U_{a_i} := U_b$ (finite subcov), $b \in \bigcap_{i=1}^{N} V_{a_i} := V_b$ open
So $A \subset U_b$, $b \in V_b$, $U_b \cap V_b = \emptyset$ $(\forall b \in B)$
Cover $B = \bigcup_{j=1}^{M} V_{b_j} = V$ (open), $A \subset U_{b_1} \cap \cdots \cap U_{b_M} = U$ (open). Then $U \cap V = \emptyset$.

(ii) Given $y_1 \neq y_2$ in Y, let $U_1 \supset f^{-1}(y_1)$, $U_2 \supset f^{-1}(y_2)$ open disjoint.
As in (i), let $V_1 = f(U_1^c)^c$, $V_2 = f(U_2^c)^c$. Note $y_1 \in V_1$, $y_2 \in V_2$
If $z \in V_1$, $f^{-1}(z) \subset U_1$ (if $f(x) = z \in f(U_1^c)^c$, $x \in U_1$)
If $z \in V_2$, $f^{-1}(z) \subset U_2$. Since $U_1 \cap U_2 = \emptyset$, it follows $V_1 \cap V_2 = \emptyset$, so
Y is <u>Hausdorff</u>.