PROBLEM SET 8: CONNECTED SPACES

1. A subset $E \subset \mathbb{R}$ is connected iff E is an interval.

2. (i) If each X_α is connected and $\bigcap_\alpha X_\alpha \neq \emptyset$, then $X = \bigcup_\alpha X_\alpha$ is connected.

(ii) If each pair of points $x, y \in X$ lie in some connected subset $E_{xy} \subset X$, then X is connected.

(iii) If $X = \bigcup_{n \geq 1} X_n$, each X_n is connected and $X_n \cap X_{n+1} \neq \emptyset\ \forall n \geq 1$, then X is connected.

3. If $E \subset X$ is connected and $E \subset A \subset \overline{E}$, then A is connected (in particular, $\overline{E}$ is connected.)

4. (i) $X = X_1 \times X_2$ is connected iff both X_i are.

(ii) $X = \Pi_\lambda X_\lambda$ is connected iff each X_λ is.

5. (i) A subset $E \subset \mathbb{R}$ is totally disconnected iff E contains no interval.

(ii) $\mathbb{Q}, \mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$ and the Cantor middle-thirds set $C \subset [0, 1]$ are totally disconnected.

Def. A *chain of open sets* connecting two points $a, b \in X$ is a sequence $U_1, \ldots, U_n$ of open sets of X such that $a \in U_1, b \in U_n$ and $U_i \cap U_j \neq \emptyset$ iff $|i - j| \leq 1$.

6. If X is connected and $\mathcal{U}$ is any open cover of X, then any two points $a, b \in X$ can be connected by a chain of open sets consisting of elements of $\mathcal{U}$.

Hint: Let Z be the set of points of X that can be connected to a by a chain of open sets in $\mathcal{U}$. It is enough to show Z is closed (since it is clearly open and non-empty.)

Pathwise connected spaces. Path connected implies connected, but not conversely. Thus the path components of X partition each cnnected component of X. Unlike the connected components, the path components need not be closed (example: 'topologists' sine curve'.)

X is *locally path connected* if for all x and all U open neighborhood of x, there exists $x \in V \subset U$ with V open and path connected.

The following conditions are equivalent:

(1) Each path component of X is open (and hence also closed, as its complement is the union of all other path components.)

(2) Each point of X has a path-connected open neighborhood.

7. If $E \subset \mathbb{R}^n$ is countable, its complement E^c is path connected.

8. A connected, locally path connected space X is path connected. In particular, connected open sets in $\mathbb{R}^n$ are path connected.

9. The unit sphere in a normed vector space (of dimension at least 2) is path connected.

Def. X is *locally connected* if each $x \in X$ has a local basis of open connected sets.

10. X is locally connected iff each component of an open set is open.

11. If $f : X \to Y$ is a continuous map, and X is connected, then its graph $\Gamma_f \subset X \times Y$ is connected. Is the converse true?

12. (i) Any collection $\mathcal{U}$ of pairwise disjoint open subsets of $\mathbb{R}^n$ is countable. (In particular, any open subset of $\mathbb{R}^n$ is the disjoint union of countably many connected components, or path components.)

(ii) If $I \subset \mathbb{R}$ is a interval and $f : I \to \mathbb{R}$ is a monotone function, the set of points of discontinuity of f is a countable subset of I.

(iii) Any open subset of $\mathbb{R}$ can be expressed (in a unique way) as a countable union of disjoint open intervals.

13. A connected metric space with more than one point is uncountable. Hence any countable metric space is totally disconnected.

14. The group $G(n)$ of $n \times n$ matrices with nonzero determinant is not connected. Its connected components are the preimages G_+, G_- of the positive (resp. negative) real numbers under the determinant function.

15. Let $f : X \to Y$ be a continuous, open, surjective map. If Y is connected and each fiber $f^{-1}(y)$ is connected, then X is connected. What about "path connected"?

16. (i) What are the components and path components of $\mathbb{R}^{\mathbb{N}}$, in the product topology?

(ii) Give $X = \mathbb{R}^{\mathbb{N}}$ the uniform topology. Show that x, y lie in the same component of X iff the sequence $x - y$ is bounded (it is enough to consider the case $y = 0$, of course.)

17. If Y is Hausdorff and connected (resp. path connected) and X is Hausdorff, is $\mathcal{F}(X; Y)$ (all maps) with the pointwise topology connected (resp. path connected?) What about $C(X, Y)$ in the compact-open topology? (Consider the case $X = S^1$, the unit circle, and $Y = \mathbb{R}^2 \setminus \{0\}$.)