MATH 447, FALL 2016–Second Problem Set. *Part 2, with Hints.*

2. [F p. 62] Show that any closed subset of a compact topological space is compact.

3.[F,p 62]. No.**7.**: In any topological space, compactness implies the Finite Intersection Property. *Hint.* For Part (a), use complements and the covering definition of compactness. Then reduce part (b) to part (a). Part (b) is known as the "Finite Intersection Property".

7. The orthogonal $n \times n$ matrices define a compact subset of R^{n^2}. (Recall an $n \times n$ matrix A is *orthogonal* if $A^T A = I_n$, where T denotes 'transpose' and I_n is the $n \times n$ identity matrix.)

Steps. (i) Note that the euclidean norm of an $n \times n$ matrix (as an element of R^{n^2}) is given (in terms of the matrix entries a_{ij}) by:

$$||A|| = [\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2]^{1/2}.$$

In particular, $||A^T|| = ||A||$ for any matrix A.

Using the Cauchy-Schwartz inequality, show that, for any two $n \times n$ matrices A, B: $||AB|| \leq ||A||||B||$.

(ii) Show that for any orthogonal $n \times n$ matrix A we have: $||A|| = \sqrt{n}$.

(iii) Use parts (i) and (ii) to show that if A is orthogonal, and B is any $n \times n$ matrix, we have the inequality:

$$||A^T A - B^T B|| \leq ||(A-B)^T A|| + ||B^T (A-B)|| \leq ||A-B||(\sqrt{n} + ||B||).$$

(iv) Let $(A_i)_{i \geq 1}$ be a sequence of orthogonal matrices converging (in norm) to an $n \times n$ matrix B. Use (iii) to show B is an orthogonal matrix.

8. Let $X \subset R^{n+1} \setminus \{0\}$ a compact set containing exactly one point on each ray from $0 \in R^{n+1}$. Show that X is homeomorphic to the unit sphere S^n.

Definition: A *ray* (from 0 in R^{n+1}) is a set of the following form (for a given $v \neq 0$ in R^n):

$$r_v = \{w \in R^{n+1} | w = \lambda v \text{ for some } \lambda \geq 0\}.$$

10. Any locally Lipschitz map $f : K \to R^n$ defined on a compact set $K \subset R^m$ is Lipschitz.

Hint: (i) Explain why, if f is not Lipschitz in K, we may find sequences x_N, y_N in K so that, for each $N \geq 1$, we have:

$$||f(x_N) - f(y_N)|| \geq N||x_N - y_N||.$$

(ii) Since f is bounded (why?), explain why we must have $||x_N - y_N|| \to 0$ as $N \to \infty$. Explain why we have subsequences x_{N_j}, y_{N_j} converging to the same limit $x_0 \in K$.

(iii) Use the fact that f is Lipschitz in some open ball $B_R(x_0)$ to show we must have, for some constant $L > 0$ (depending on R and x_0):

$$||f(x_{N_j}) - f(y_{N_j})|| \leq L||x_{N_j} - y_{N_j}||,$$

for all j sufficiently large. Explain why this contradicts part (i).