Heat kernel in R^n. (Solution of $u_t = \Delta u$):

$$p(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \quad x \in R^n, t > 0.$$

1. (i) Solve the heat equation on R with initial data $u_0(x) = e^{ax}$. *Ans.* e^{ax+a^2t} (ii) Find all solutions of the heat equation on R that evolve by multiplication: $u(x,t) = f(t)u_0(x), f(0) = 1$.

2. Solve the heat equation $u_t = u_{xx}$ on R with initial condition:

$$u_0(x) = 3, |x| \leq 1; \quad u_0(x) = 1, |x| > 1.$$

(i) Express your solution in terms of the error function:

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz.$$

Recall $\frac{1}{2} + \frac{1}{2} Erf(\frac{x}{\sqrt{4t}})$ is the solution with initial data $\sigma(x)$, the unit step function at 0.

(ii) Find $\lim_{t\to\infty} u(x,t)$.

3.(i) Solve the heat equation $u_t = u_{xx}$ with initial data $u_0(x) = x^2$, by the following method: the third derivative u_{xxx} is also a solution, with initial data zero. Hence $u_{xxx}(x,t) \equiv 0$, so:

$$u(x,t) = A(t)x^2 + B(t)x + C(t).$$

Impose the heat equation to find O.D.E. for $A(t), B(t), C(t)$, and solve them. (Clearly this generalizes to any polynomial initial data).

(ii) Use part (i) and the heat kernel to find the value of the integral $\int_{-\infty}^{\infty} z^2 e^{-z^2} dz$.

(iii) Use the same method to solve the heat equation in R^3 with initial data $u_0(x,y,z) = xy^2z$.

4. (*Method of images* for the heat kernel). The solution of the Dirichlet problem for the heat equation in the upper half-plane $H = \{(x_1,x_2) \in R^2; x_2 > 0\}$:

$$u_t = \Delta u, \quad u(x,t) = 0 \text{ for } x \in \partial H \forall t > 0, \quad u(x,0) = u_0(x)$$

is given by:

$$u(x,t) = \int_H p_H(x,y,t) u_0(y) dy,$$

where $p_H(x,y,t)$, for $x \in H, y \in \bar{H}$ (the closed half-plane) is defined by the properties: (i) $p_H(x,y,t) \sim p(x-y,t)$ for x close to y; (ii) for $y \in \bar{H}$ fixed, $p_H(x,y,t)$ is a solution of the heat equation, $x \in H \setminus \{y\}, t > 0$; (iii) $p_H(x,y,t) = 0, \forall x \in H, y \in \partial H, t > 0$.

Find an explicit expression for $p_H(x,y,t)$ using the method of images. (Of course this could be done for the disk as well).

Kirchhoff's formula. Solution of the wave equation in R^3:

$$u_{tt} = \Delta u, \quad u(x,0) = u_0(x), u_t(x,0) = u_1(x).$$

$$u(x,t) = t\text{Ave}[u_1; S(x,t)] + \frac{\partial}{\partial t}(t\text{Ave}[u_0; S(x,t)]),$$

where $\text{Ave}[v; S(x,R)] = \frac{1}{4\pi R^2}\int_{S(x,R)} v dA$ denotes the average value of the function v over the sphere of center x, radius R.

5. Solve with $u_0 \equiv 0, u_1(x,y,z) = y$.

6. Solve with $u_0 \equiv 0, u_1(x) = |x|^2$.

8. Solve with $u_0 \equiv 0, u_1 = A$ for $|x| < \rho, u_1 = 0$ for $|x| > \rho$. Where is the solution nonzero?

Remark 1. We'll need the answer to the following geometric question: what is the area of the intersection $S(x,R) \cap \{|x| < \rho\}$? *Answer:*

$$(\pi R/|x|)(\rho^2 - (|x| - R)^2) \text{ if } |\rho - R| \leq |x| \leq \rho + R.$$

Remark 2. From the answer we easily obtain the solution of the same problem with u_0 and u_1 interchanged.