THE ONE-DIMENSIONAL WAVE EQUATION

1. The one-dimensional linear wave equation (WE) on the real line is:

$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t), x \in \mathbb{R}, t > 0.$$

In fact we could consider negative values of t as well, which is sometimes useful. Here $c > 0$ is a constant.

It is easy to check that, if F is a C^2 function defined on $\mathbb{R}$, $u(x,t) = F(x-ct)$ and $u(x,t) = F(x+ct)$ are solutions of the WE. They describe the "disturbance" $u(x,0) = F(x)$ moving by translation to the right (resp. to the left) with constant speed c, without distortion. (That's why it's called the "wave equation".) Since the equation is linear homogeneous (sums and constant multiples of solutions are solutions), given arbitrary functions F, G in $C^2(\mathbb{R})$ the function:

$$u(x,t) = F(x-ct) + G(x+ct), \quad x \in \mathbb{R}, t > 0$$

is also a solution of the WE. A good question is: *are all solutions of this form?*

To answer this we appeal to two far-reaching ideas. The first is to use the "differential operators" ∂_x and ∂_t (partial derivatives in x and t) to write the wave equation in the form:

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0.$$

(This is easy to verify, since for instance $\partial_x \partial_x u = u_{xx}$, and should be reminiscent of the algebraic identity $a^2 - b^2 = (a-b)(a+b)$.)

So a solution u of the wave equation gives rise to solutions u and p of two first-order problems:

$$(\partial_t - c\partial_x)p = 0, \quad (\partial_t + c\partial_x)u = p.$$

The second idea is to observe these first-order differential operators can be thought of as first-order partial derivatives with respect to new variables, referred to in Physics as 'advanced/retarded' variables:

$$z_+ = x + ct, \quad z_- = x - ct, \quad x = \frac{1}{2}(z_+ + z_-), t = \frac{1}{2c}(z_+ - z_-),$$

so that:

$$(\partial_t + c\partial_x)f = 2c\partial_{z_+} f, \quad (\partial_t - c\partial_x)f = -2c\partial_{z_-} f.$$

Exercise 1: verify this, using the chain rule for functions of two variables.

Thus the two problems above can be written as (for functions $p(z_+, z_-), u(z_+, z_-)$):

$$\partial_{z_-} p = 0, \quad 2c\partial_{z_+} u = p.$$

The first problem has the general solution:

$$p(z_+, z_-) = c(z_+), \text{ an arbitrary function of } z_+.$$

By integration, the second problem has the general solution (for an antiderivative C of c, as a function of z_+):

$$u(z_+, z_-) = \frac{1}{2c}C(z_+) + q(z_-),$$

where we think of q as a "constant of integration", depending on z_-. Writing this in terms of the original variables (x, t):

$$u(x, t) = p(x + ct) + q(x - ct), \text{ where } p = \frac{1}{2c}C$$

answering the question in the second paragraph.

2. Cauchy problem. Physically the WE describes small oscillations of a "string" (one-dimensional elastic continuum), subject only to the force "tension" (which is tangential to the graph of $u(., t)$ at each point). Since $u_{xx}(x, t)$ can be thought of as the "curvature" at x of the graph of $u(\cdot, t)$ (the solution at time t), writing the WE in the form $u_{tt} = -c^2 u_{xx}$, we see that the string at each fixed x is moving "up and down" with acceleration proportional to minus the curvature, accelerating where the graph of $u(\cdot, t)$ is *concave*, slowing down where it is *convex*. The equation is second order in time, so our experience with ordinary differential equations (and Mechanics) suggests two initial conditions–the initial position $u(x, 0)$ and initial velocity $u_t(x, 0)$– should completely determine the subsequent motion of the string. That is, the following initial-value problem for the WE (known as the *Cauchy problem*) is physically reasonable:

$$u_{tt} - c^2 u_{xx} = 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R}.$$

Here f and g are given smooth functions. Looking for a solution of the form:

$$u(x, t) = p(x + ct) + q(x - ct)$$

we find a system of first-order differential equations for p, q, easily solved to give p, q in terms of f, g:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)du.$$

This important expression is known as *d'Alembert's formula.* Letting G denote the anitderivative of g vanishing at $x = 0$, we may write this as:

$$u(x,t) = \frac{1}{2}[(f + \frac{1}{c}G)(x+ct) + (f - \frac{1}{c}G)(x-ct)]], \quad G' = g, G(0) = 0.$$

This exhibits u as the sum of a right-moving wave and a left-moving one, both with speed c. And introducing the interval $I_x(ct) = [x - ct, x + ct]$ we may also write the solution in the form:

$$u(x,t) = \text{ave}[f; x \pm ct] + t\text{ave}[g; I_x(ct)];$$

That is, to find the value of the solution at x at time $t > 0$, compute t times the average value of the initial velocity on the interval $I_x(ct)$ with center x, radius ct, then the average value of the initial position on the boundary of this interval and add them together.

2.1 *Estimates.* We can use this last interpretation to say something quantitative about the "size" (maximum amplitude) of the solution. For a function f bounded over the whole real line (that is, there exists an $M > 0$ so that $|f(x)| \leq M$ for all $x \in \mathbb{R}$) , define the norm of f (denoted by $||f||$) as the smallest $M > 0$ for which this is true. Then if the initial data f and g are both bounded functions on $\mathbb{R}$, so is the solution at each time $t > 0$, with the estimate:

$$||u(\cdot,t)|| \leq ||f|| + t||g||.$$

This shows that if the initial velocity is non-zero, the range of motion (at each point) may grow up to linearly in time. For instance, it is easy to check that d'Alembert's formula gives:

$$f(x) \equiv 0, g(x) \equiv 1 \Rightarrow u(x,t) = t, x \in \mathbb{R}, t > 0.$$

(The infinite string just moves upwards as a single straight line, with constant speed one. The WE does admit very boring solutions.)

2.2 *Supports.* Suppose a function on the real line is identically zero outside of some interval. Then the closure of the largest open set where the function is *not* zero is called its *support*. For example, the support of the function:

$$f(x) = 1 - |x|, |x| \leq 1; f(x) = 0, |x| \geq 1$$

is the interval $[-1, 1]$. If the support of a function is a bounded set we say it has "compact support". The one-dimensional WE has the following important property:
If the initial data f, g have compact support, so does the solution $u(\cdot, t)$, for each t.
To see this, suppose (to simplify things) the support of both f and g is the interval $[-R, R]$. Then from d'Alembert's formula we see that:

$$u(x,t) \neq 0 \Leftrightarrow I_x(ct) \cap [-R, R] \neq \emptyset,$$

so on the intervals $\{x > R + ct\}$ and $\{x < -R - ct\}$ the solution $u(\cdot, t)$ vanishes identically. Another way to say this is that the support of $u(\cdot, t)$ is contained in (in fact *equals*) the closed interval $[-R - ct, R + ct]$: the size of the support grows linearly with t. Physically this says "disturbances" from equilibrium proagate on the real line with speed c.

Another physically interesting way to see this is this is to consider the support of $u(x, \cdot)$, the solution as a function of $t > 0$, for fixed x. Suppose the support of the initial data $\{f, g\}$ is $[-R, R]$, and the observer (you) stands at $x \in \mathbb{R}$, where $x > R$. Then you don't feel the disturbance until it reaches you at $t = \frac{1}{c}(x - R)$, then for a while you do (precisely, for $2R/c$ time units), then you don't feel it anymore:

$$u(x,t) = 0 \text{ for } t < \frac{1}{c}(x - R) \text{ or } t > \frac{1}{c}(x + R).$$

Stated mathematically, as a function of t for fixed x the support of the solution is the closed time interval $[\frac{1}{c}(x - R), \frac{1}{c}(x + R)]$, if $x > R$.

Exercise 2.(i) Sketch a smooth, even positive 'bump function' with maximum value 1 (at $x = 0$) and support $[-1/2, 1/2]$. Use this as $u(x, 0)$ (and $u_t(x, 0) \equiv 0$) to define a solution of the WE with $c = 1$. (ii) What is the earliest time the graph of $u(x, t)$ (as a function of x) shows two separate 'bump functions' identical to the original one, moving to the right and to the left? Sketch the graph of $u(x, 2)$. (iii) Now fix $x_0 = 3$, and sketch the graph of $u(3, t)$ as a function of $t > 0$.

2.3 *Singularities and characteristics.* If f is a C^2 function and g is C^1, the solution of the Cauchy problem given by d'Alembert's formula is C^2

in spacetime, so the second partial derivatives u_{xx} and u_{tt} are continuous functions. But it is easy to see that the f part of the formula makes sense for *any* function f defined on the real line, while the g part certainly makes sense if g is piecewise continuous. But then u_{xx} and u_{tt} will not be defined everywhere.

Sometimes it is useful to consider functions that are not of class C^2 everywhere on $\mathbb{R}$; let's say $x \in \mathbb{R}$ is a *singular point* of f if f is not of class C^2 in any neighborhood of x, but has finite one-sided limits at x. Then d'Alembert's formula still makes sense if the sets of singular points of f and g are discrete (that is, are finite in any bounded interval). It generates a solution $u(x,t)$ which is C^2 in a certain open set of spacetime (the "regular set") and solves the equation there.

What is the set of singular points of $u(.,t)$ for $t > 0$? From d'Alembert's formula, we see that if f is singular at x_0, $u(\cdot,t)$ will be singular at the points $x_0 \pm ct$. The lines $x = x_0 \pm ct$ are called *characteristic lines* from $(x_0, 0)$. Likewise, each singular point x_0 of G (the antiderivative of g) generates two singular points $x_0 \pm ct$ for $u(\cdot,t)$. (But note that $f + \frac{1}{c}G$ may have fewer singular points than f or G in exceptional cases). We say "singularities of the data propagate (to the solution) along characteristic lines".

Example. Let $f(x) = 1 - |x|$ for $|x| \leq 1$, $f(x) = 0$ elsewhere (assume $g = 0$). The singular set of f is $\{-1, 0, 1\}$, and if we set $u(x,0) = f(x), u_t(x,0) = 0$, the singular set of the solution at time t has six points (in general) $\{-1 \pm ct, \pm ct, 1 \pm ct\}$ (some of these may coincide for isolated values of t. The singular set of solution in spacetime is consists of the characteristic lines $x = -1 \pm ct, x = \pm ct, x = 1 \pm ct$.

Exercise 3. In a spacetime diagram (for $t > 0$) sketch the characteristic lines along which the singularities are propagating in this example (use $c = 1$.) Then sketch the graphs of $u(x, 3/2)$ and $u(x, 5/2)$.

2.4 *Sets of dependence and of influence.* The value of d'Alembert's solution at (x,t) depends only on the values of the initial data $\{f, g\}$ on the interval $I_x(ct) = [x - ct, x + ct]$, the *interval of dependence* of (x,t). The value of the initial data near the point x_0 affects the solution at time t only near the *points* $x_0 \pm ct$ the *set of influence* of the x_0 at time t.

3. Nonhomogeneous wave equation. The motion of a string under the action of a time-dependent external force (in addition to the tension) is described by the wave equation with a 'force term' $f(x,t)$, the *nonhomoge-*

neous wave equation:

$$u_{tt} - c^2 u_{xx} = f(x,t), x \in \mathbb{R}, t > 0.$$

We consider this problem first with zero initial conditions. Then the d'Alembert solution is given by a double integral:

$$u(x,t) = \frac{1}{2c} \int \int_{\Delta^{(x,t)}} f(y,s) dy ds, \text{ where } \Delta^{(x,t)} = \{(y,s); 0 \leq s \leq t, x - c(t-s) \leq y \leq x + c(t-s)\}$$

is the "backwards light cone" at (x,t). Slicing the triangle by horizontal segments, we may write:

$$u(x,t) = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s) dy ds.$$

Note that this expression defines a function of class C^2 in (x,t), assuming only f continuous in (x,t). It is an excelent *calculus exercise* to verify this function does solve the non-homogeneous WE.

If we now consider the problem with non-zero initial data $u(x,0) = u_0(x), u_t(x,0) = u_1(x)$, we have d'Alembert's formula for the non-homogeneous problem:

$$u(x,t) = \frac{1}{2}[u_0(x+ct) + u_0(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} u_1(s) ds + \int \int_{\Delta^{(x,t)}} f(y,s) dy ds.$$

3.1 *Estimates.* Note that, since the area of the "backwards light cone" is ct^2, we may write the d'Alembert solution of the nonhomogeneous WE (with zero initial data) in the form:

$$u(x,t) = \frac{t^2}{2} \text{ave}[f; \Delta^{(x,t)}].$$

Thus if f is bounded in space time, with norm $||f||$, we have the estimate:

$$||u(\cdot,t)|| \leq \frac{t^2}{2} ||f||.$$

This shows the solution is bounded for each t, but the bound increases up to quadratically as a function of t. For the problem with nonzero initial conditions, combining with 2.1 we find:

$$||u(\cdot,t)|| \leq ||u_0|| + t||u_1|| + \frac{t^2}{2} ||f||.$$

3.2 *Supports.* Suppose the external force is present only for a bounded time interval, and that for each of these times it acts only on a bounded portion of the string. Then the support of f is contained in a set of the form:

$$Q = \{(x,t); t_0 \leq t \leq t_1, |x - x_0| \leq R\}, \text{ where } t_1 > t_0 \geq 0, x_0 \in \mathbb{R}, R > 0.$$

Equivalently, $f(x,t) = 0$ if $(x,t) \notin Q$. For which (x,t) does the backward light cone $\Delta^{(x,t)}$ from (x,t) have non-empty intersection with Q? Answering this geometric question we find the support of the solution $u(x,t)$ is contained in the set: intersect Δ, we find that $u(x,t)$ (the d'Alembert solution with force term f and zero initial data) is zero for (x,t) outside of the closed set:

$$K = \{(x,t); t \geq t_0 |x - x_0| \leq R + c(t - t_0)\},$$

the domain of influence of the interval $[x_0 - R, x_0 + R]$ starting at at $t = t_0$. In general the solution is *not* zero at points inside K. Thus we see that, although the solution $u(\cdot, t)$ has compact support for each $t > 0$, it never becomes identically zero, no matter how large t is (unlike the external force f): the effect of an external force acting for a bounded amount of time persists forever (and the portion of the string that 'feels' the effect of the force increases with time).

We can also consider the case when the external force is constant in time, and has compact support as a function of x. Say the support of $f(x)$ is contained in $[x_0 - R, x_0 + R]$.

Exercise. Show the support of the solution is a set similar to K above; but with $T_0 = 0$. Thus the solution has compact support for each t, but is never identically zero (unless $f \equiv 0$.)

4. The characteristic parallelogram property. Solutions of the homogeneous wave equation (in one space dimension) have an amazing and very useful property. Consider a parallelogram in the (x,t) half-plane ($t \geq 0$) with sides parallel to the characteristic directions (lines with slope $\pm 1/c$.) Say the vertices are:

$$A(x,t), B(x + a, t + \frac{a}{c}), D(x + b, t - \frac{b}{c}), C(x + a + b, t + \frac{a - b}{c}).$$

Then we have:

$$u(A) + u(C) = u(B) + u(D).$$

The sums of the values of the solutions on opposite vertex-pairs of a characteristic parallelogram coincide.

Exercise. Verify this, assuming the solution has the form $F(x+ct)+G(x-ct)$. (We will soon see these are all the solutions.)

This property is very useful to solve boundary-value problems, to which we turn next.

5. Boundary-value problems on the half-line. For wave propagation on a semi-infinite string $\{x>0\}$, we need to specify what happens at the boundary.

5.1 We first consider the homogeneous *Dirichlet problem*, where the string is held fixed at $x=0$:

$$u_{tt}-c^2u_{xx}=0 \quad x\geq 0, t>0, u(0,t)=0, u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\geq 0$$

Note the initial data must satisfy *compatibility conditions*: $u_0(0)=0, u_1(0)=0$. If these conditions don't hold, we don't expect a smooth solution.

Technical point: Even more strongly, if we want u to have second derivatives in x and t continuous in the *closed* quarter-plane $x\geq 0, t\geq 0$, there is a further compatibility condition, namely $u_0''(0_+)=0$. To see this, note the equation implies $u_{xx}=0$ if $u_{tt}=0$, and the boundary condition implies $u_{tt}(0,0)=0$ (under the continuity assumption on second derivatives.) If we don't require this, the discontinuity in the second derivative will propagate (along a characteristic) into the *interior* of the (x,t) quadrant.

Assume u_0 and u_1 vanish at $x=0$, and their derivatives have one-sided limits at 0_+. Then we can extend them as C^1 *odd* functions $\bar{u_0}, \bar{u_1}$ to the real line (and even C^2, if they satisfy the further compatibility condition just mentioned.) Applying d'Alember's formula to the extended $\bar{u_0}, \bar{u_1}$ we find a solution of the wave equation on $x\in\mathbb{R}, t>0$ (possibly not C^2 everywhere) satisfying the given initial conditions where $x>0$.

Exercise. Verify that the d'Alembert solution is an odd function of x.

Since the solution obtained is (at least) continuous (and odd under $x\mapsto -x$ by the exercise), it follows the boundary condition $u(0,t)=0$ holds for all t. It is easy and useful to write an expression for the solution that depends only on the initial data as given (on the half-line). There are two cases: (i) If $x\geq ct$, d'Alembert's formula involves only the initial data on the set $\{x\geq 0\}$, and the expressions for $u(x,t)$ look exactly as in the whole-line case. (ii) If $x \leq ct$, we have (*exercise!*):

$$u(x,t) = \frac{1}{2}[u_0(x+ct) - u_0(ct-x)] + \frac{1}{2c}[U_1(x+ct) - U_1(ct-x)],$$

where U_1 is any antiderivative of u_1.

Supports and interpretation. To get an idea of what these expressions are saying, suppose $u_1 \equiv 0$, and u_0 is a smooth positive 'bump' supported in a small interval around a point $x_0 > 0$. If we "run the movie" expressed by these formulas, what do we see? For $t > 0$ small (that is, $t < x_0/c$) the 'bump' splits into two packets (as usual), one propagating to the right, the other to the left, both with speed c. (Note: in fact to actually see *two* separate packets we have to wait roughly $2\delta/c$ time units, where 2δ is the length of the support of u_0.) There are also two *inverted*"virtual packets" moving behind the wall $x = 0$, located at $-x_0 - ct$ (moving left) and at $-x_0 + ct$ (moving right)

Then at about $t = x_0/c$ the left-moving physical packet 'hits the wall' $x = 0$, and something happens: it starts being cancelled by the right moving virtual packet, which suddenly crosses into the physical region $\{x > 0\}$. Viewed from the physical region, the packet appears to be gradually absorbed by the wall, and emitted back "inverted". If we wait a little longer (for $t > (x_0 + 3\delta)/c$, say), we see two packets moving to the right, the original right-moving one (which by now is at a distance about $2x_0$ from the wall), and an inverted packet close to the origin. Both packets then continue to move to the right with constant speed, seprated by a distance $2x_0$. We can describe what happened as "reflection with inversion" of the left-moving packet.

Exercise 4. Draw a diagram illustrating this, showing the two packets at time moments before and after the reflection-inversion.

Specifically, let u_0 be a 'positive bump' with peak value 1 at $x_0 = 3$ and support $[3-1/2, 3+1/2]$, and let $u_1 \equiv 0$. Evolve u_0 by the WE with $c = 1$ on the half-line $x > 0$, with Dirichlet boundary conditions. Sketch the graphs of $u_0(x), u(x,2), u(x,3), u(x,5)$. Include the 'virtual packets' (located on the negative half-line $\{x < 0\}$) in your diagrams.

5.2 Now consider the *Neumann problem*, where there is no force at the endpoint $x = 0$ of the string, so in particular the vertical component of the tension is zero, so $u_x = 0$. The Cauchy problem is:

$$u_{tt} - c^2 u_{xx} = 0, x > 0, t > 0; u_x(0,t) = 0, \quad u(x,0) = u_0(x), u_t(x,0) = u_1(x).$$

Again we have *compatibility conditions:* $u_0'(0) = 0$ and $u_1'(0) = 0$ if we want the solution to be smooth. (In addition, we want u_{0xx} to have a finite one-sided limit as $x \to 0_+$.)

The idea now is to extend u_0, u_1 to the real line as *even* functions $\bar{u_0}$ and $\bar{u_1}$ (of classes $C^2(\mathbb{R}$ and $C^1(\mathbb{R})$, respectively.) Just set $\bar{u_0}(x) = u_0(-x)$ and $\bar{u_1}(x) = u_1(-x)$ for $x < 0$. Then define a solution $u(x,t)$ of the WE on $x \in \mathbb{R}, t > 0$ by applying d'Alembert's formula to $\bar{u_0}$ and $\bar{u_1}$.

Exercise: Verify that the solution given by d'Alembert's formula is C^2 on the whole half-plane $\{t > 0\}$ and an *even* function of x. In particular, it satisfies the boundary condition $u_x(0,t) = 0$ for all t.

As in the previous case, the solution can be written directly in terms of the initial data, as defined for $x > 0$. For $x \geq ct$, the expression is the same as in the whole-line case. For $x \leq ct$, we have:

$$u(x,t) = \frac{1}{2}[u_0(x+ct) + u_0(ct-x)] + \frac{1}{2c}[U_1(x+ct) + U_1(ct-x)],$$

where U_1 is the antiderivative of u_1 vanishing at $x = 0$.

Support and interpretation. As before, we consider the motion of a small positive smooth 'bump' $u_0(x)$, with support in a small interval centered at $x_0 > 0$ (and set $u_1(x) = 0$). Initially (for $t < x_0/C$), after a short time we observe two identical 'bumps' (each half as tall as the original one), one moving to the right, the other to the left, both with constant speed c. Behind the 'wall' $x = 0$ we can imagine two 'virtual bumps' (at $-x_0 - ct$, moving to the left, and at $-x_0 + ct$, moving to the right); this time not 'inverted', but identical with the 'physical' bumps. At about $t = x_0/c$, the left-moving physical bump hits the wall, simultaneously with the right-moving virtual bump. On the 'physical side' $x > 0$, we see the 'bump' grow back to its original size as it hits the wall, then start moving to the right as it regains its half size. Finally we see two bumps, both identical to the original one (but half as tall), moving to the right at constant speed c, separated by a constant distance (about $2x_0$): we have *reflection without inversion.*

Remark: One can treat the non-homogeneous wave equation $u_{tt} - c^2 u_{xx} = f$ with Dirichlet or Neumann boundary conditions by the same method: extending the force function f and the initial data to the region $x < 0$ as odd (resp. even) functions. The details are left as an exercise for the reader.

5.3. *Non-homogeneous boundary conditions.* Consider first the *Dirichlet case* , with zero initial conditions. (By linearity, it is enough to consider this case.)

$$u_{tt} - c^2 u_{xx} = 0, x > 0, t > 0; \quad u(0,t) = h(t), t \geq 0; u(x,0) \equiv 0, u_t(x,0) \equiv 0.$$

There are *compatibility conditions:* $h(0) = 0, h'(0) = 0$. In addition, if $u_{tt} = c^2 u_{xx}$ is to extend continuously to the closed first quadrant $\{t \geq 0, x \geq 0\}$, we must have $h''(0) = 0$.

Assume a solution exists. Then by d'Alembert's formula, $u(x,t) \equiv 0$ in the region $\{(x,t); x \geq ct\}$.Now let (x,t) satisfy $0 < x \leq ct$, and recall the characteristic parallelogram property from 4. above. We find a characteristic parallelogram with vertices (x,t), $(0, t - x/c)$ (connected by a characteristic segment) and two points on the line $x = ct$. We conclude:

$$u(x,t) = h(t - \frac{x}{c}), \quad t \geq x/c; u(x,t) = 0, x \geq ct.$$

Exercise. Verify that this defines a C^2 function of (x,t) in the first quadrant, which solves the equation and the boundary and initial conditions.

Support and interpretation. Suppose $h(t)$ is positive in $(t_0 - \delta, t_0 + \delta)$, and zero outside this interval. (So h is a 'pulse' that happens at time t_0, at the origin. Then we can consider $u(\cdot, t)$, the solution at time t. This is conveniently done sketching a spacetime diagram, with the characteristic lines $x = c(t - t_0) \pm c\delta$ issuing from the points $(0, t_0 - \delta)$ and $(0, t_0 + \delta)$. The support of the solution is contained between these lines, and we consider three ranges for t:
1) If $t \leq t_0 = \delta$, $u(x,t) \equiv 0$.
2) If $t_0 - \delta < t < t_0 + \delta$, then $u(x,t)$ is positive for $0 < x < c(t - t_0) + c\delta$ and zero for $x \geq c(t - t_0)$.
3) If $t > t_0 + \delta$, $u(x,t) = 0$ if $0 < x \leq c(t - t_0) - c\delta$ or if $x \geq c(t - t_0) + c\delta$.

Thus the solution has compact support for each $t > 0$, and at time t consists of a 'pulse' located at $c(t - t_0)$: the 'pulse' travels to the right with speed c, without distortion.

Exercise Verify these assertions using a spacetime diagram.

We can also consider $u(x_0, t)$, the solution as a function of t, as viewed by an observer at a point x_0. It is easy to see that $u(x_0, t) = 0$ if $t \leq \frac{x_0}{c} + t_0 - \delta$ or $t \geq \frac{x_0}{c} + t_0 + \delta$, and $u(x_0, t)$ is positive if $(\frac{x_0}{c} + t_0 - \delta, \frac{x_0}{c} + t_0 + \delta)$. The observer experiences a pulse of duration 2δ, at time $\frac{x}{c} + t_0$ (the pulse is emitted at the origin, at time t_0, and it takes $\frac{x_0}{c}$ time units to arrive at position x_0). *Verify this!*

The case of *Neumann* boundary condition ($u_x(0,t) = g(t)$) is similar.

$$u_{tt} - c^2 u_{xx} = 0, x > 0, \quad u_x(0,t) = g(t), \quad u(x,0) = 0, u_t(x,0) = 0.$$

The *compatibility conditions* are $g(0) = g'(0) = 0$, and it is convenient to extend $g(t)$ as zero for $t \leq 0$. To guess the form of the solution, note that if $u(x,t)$ solves this problem, then $v(x,t) = u_x(x,t)$ solves the Dirichlet problem on the half-line, with boundary condition $g(t)$ (the problem just discussed). So we should have:

$$u_x(x,t) = g(t - \frac{x}{c}), t \geq 0, x \geq 0.$$

(Recall we extend g as zero for negative values of the argument.) Then $u(x,t)$ may be found by integration in space:

$$u(x,t) = u(0,t) - c\int_0^x g(t - \frac{s}{c})ds = u(0,t) - c^2 \int_{t-\frac{s}{c}}^t g(u)du$$

(making the change of variable $u = t - s/c$.)

Can $u(0,t)$ be chosen arbitrarily? We suspect not, and proceed to verify if the above expression does solve the equation. Differentiating in x and t, we find (*exercise!*):

$$u_t(x,t) = u_t(0,t) - c^2(g(t) - g(t - \frac{x}{c})), \quad u_{tt}(x,t) = c^2 g'(t - \frac{x}{c}) + u_{tt}(0,t) - c^2 g'(t),$$

$$u_x(x,t) = -cg(t - \frac{x}{c}), \quad u_{xx} = g'(t - \frac{x}{c}).$$

So if u is a solution, we must have:

$$u_{tt}(0,t) = c^2 g'(t), \quad u_t(0,t) = c^2 g(t), \quad u(0,t) = c^2 G(t),$$

where G is the antiderivative of g satisfying $G(0) = 0$. This simplifies to:

$$u(x,t) = c^2 G(t) - c^2(G(t) - G(t - \frac{x}{c}) = c^2 G(t - \frac{x}{c}),$$

a convenient form of the solution. Note that if g has compact support (say, in $[t_0 - \delta, t_0 + \delta]$) *and* zero average ($\int_{t_0-\delta}^{t_0+\delta} g(u)du = 0$, then G also has compact support (in the same interval), and the analysis of the support of solutions described earlier for the Dirichlet case still applies.

7. Problems on a bounded interval. The small oscillations of a finite string (subject only to the tension force) are described by the homogeneous wave equation on an interval $[0, L]$ with boundary conditions.These lead to compatibility conditions that must be satisfied by the initial data $u(x, 0) = u_0(x), u_t(x, 0) = u_1(x)$. The most typical ones are:

(i) *Dirichlet:* $u(0, t) = u(L, t) = 0$, for all t. The string's endpoints are held fixed, by an external force acting only on the boundary. The compatibility conditions are: $u_0(0) = u_0(L) = 0$, $u_1(0) = u_1(L) = 0$.

(ii) *Neumann:* $u_x(0, t) = 0, u_x(L, t) = 0$, for all t. The string's endpoints move freely, so the vertical component of the tension (which is proportional to u_x) must vanish at the endpoints. The initial data must also satisfy $u_{0x}(0_+) = u_{0x}(L-) = 0$, $u_{1x}(0_+) = u_{1x}(L-) = 0$.

(iii)*Periodic:* $u(0, t) = u(L, t)$ and $u_x(0, t) = u_x(L, t)$. In this case the motion may be extended to the whole real line, as a periodic function with period L. We typically require also $u_0(x) = u_0(L), u_{0x}(0) = u_{0x}(L), u_1(0) = u_1(L), u_{1x}(0) = u_{1x}(L)$.

A convenient method to solve these boundary-value problems is to extend the initial data to the whole real line (as we did earlier for problems on the half-line). The type of extension is dictated by the boundary conditions (BC), as follows:

(i) For Dirichlet BC, extend u_0 and u_1 to the real line as *odd* functions, both with respect to the usual reflection $x \mapsto -x$ and the reflection $x \mapsto 2L - x$, on the line $x = L$. Equivalently, require the extension to be odd (in the usual sense) and $2L$-periodic. (In fact, any two of these conditions imply the third one; *check this!*)

Now find the solution $u(x, t)$ by applying d'Alembert's formula to the extended initial data u_0, u_1. The part of the solution corresponding to u_0 is clearly odd and $2L$-periodic. The part corresponding to u_1:

$$\frac{1}{2c}\int_{x-ct}^{x+ct} u_1(s)ds = \frac{1}{2c}[U_1(x + ct) - U_1(x - ct)],$$

where U_1 is an antiderivative of u_1. Since U_1 is even, we see easily that this is an odd function of x. And since the integral of u_1 on any interval of length $2L$ has the same value (namely zero, since this is the integral of u_1 from $-L$ to L) we again see easily that this expression is $2L$-periodic. Thus the solution $u(x, t)$ satisfies the Dirichlet boundary conditions at 0 and at L.

For Neumann BC, extend u_0 and u_1 to the real line as even functions of x, and $2L$-periodic. As a consequence, u_0 and u_1 will also be even with respect to the reflection $x \mapsto 2L - x$. Again apply d'Alembert's formula to the extended initial data. We leave it to the reader to verify this solution does satisfy the boundary conditions.

Exercise. (i) Show that the solution given by d'Alembert's formula defines a function $u(x,t)$ which is even (in the usual sense, under $x \mapsto -x$) and $2L$-periodic.

(ii) Show that if a function $f(x)$ is even with respect to reflection on $x = L$ (that is, $f(2L - x) = f(x)$ for all x) and everywhere differentiable, then $f'(L) = 0$. In particular, the solution found above does satisfy the Neumann BC $u_x(0,t) = u_x(L,t) = 0$.

For the problem with periodic boundary conditions, we (naturally enough) extend the initial data u_0, u_1 to the real line as L-periodic functions (which won't necessarily be smooth, but will at least be of class C^1.) then again apply d'Alembert's formula. A slight problem is that, although the u_0 part of the solution is clearly L-periodic, the u_1 part is not periodic, unless we happen to know that $\int_0^L u_1(s)ds = 0$. If this isn't the case, write $u_1 = \tilde{u_1} + C$, where $\tilde{u_1}$ has zero integral over $[0,L]$. Then apply d'Alembert's formula to $u_0, \tilde{u}_1$ to get the solution $\tilde{u}(x,t)$, which is easily seen to be L-periodic. Now let $u(x,t) = Ct + \tilde{u}(x,t)$. One easily checks (do it) this function is L-periodic, as desired, and satisfies the initial conditions.

Remark 1. Non-homogeneous wave equation in $[0,L]$. The same method can be used to solve the nonhomogeneous WE $u_{tt} - c^2 u_{xx} = f(x,t)$ on an interval $[0,L]$, with Dirichlet, Neumann or periodic boundary conditions. Now extend both the initial conditions u_0, u_1 and the force term f as odd, $2L$-periodic functions of x (resp. even, $2L$-periodic, resp. L-periodic.) In general the extended $f(x,t)$ will only be piecewise continuous, but the d'Alembert solution can still be found in this case. It is then not hard to show (as before) the resulting $u(x,t)$ satisfies the desired boundary condition. The details are left to the reader.

Remark 2. Periodicity in time. For Dirichlet boundary conditions the d'Alembert solution is periodic in time, with period $2L/c$. For the Neumann problem, this is still true *provided* the initial velocity u_1 has zero integral over $[0,L]$: $\int_0^L u_1(s)ds = 0$. In general, it is the sum of a $2L/c$-periodic function (in time) and one of the form Ct, for a suitable C. A similar consideration holds for the problem with periodic BC. Also for the non-homogeneous problem with force term depending only on x ($f = f(x)$), we again find that the solution is $2L/c$-periodic in time (up to adding a linear function of t, in the Neumann case.)

Remark 3. Graphical solution of the Dirichlet problem. The non-homogeneous Dirichlet problem for the homogeneous WE on an interval $[0, L]$ (say, with zero initial conditions):

$$u_{tt} - c^2 u_{xx} = 0, u(0,t) = h(t), u(L,t) = g(L), \quad u(x,0) = 0, u_1(x,0) = 0.$$

can be solved "graphically", using the paralleloram property. Namely, we divide $[0, L] \times [0, L/c]$ into four regions: on I the solution is zero, while any point in region II (or region (III) is a vertex of a parallelogram with one vertex on the $x = 0$ axis, and two on a characteristic line on the boundary of region I, where $u = 0$. Now continue in a similar way to region IV. This method is very convenient if one wants the solution at a given point, but it sheds no insight on the form of the solution (say, as a function of x for each fixed t).

Exercise 5. Find expressions for the value of the solution $u(x,t)$, for (x,t) in regions II,III and IV, using the characteristic parallelogram property.

8. The energy and uniqueness.

Let $u(x,t)$ be a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$. The expressions:

$$e(x,t) = \frac{1}{2}(u_t^2 + c^2 u_x^2), \quad p(x,t) = -c^2 u_x u_t$$

are often called "energy density" and "momentum density" of the solution. The integral of e over an interval $I = [a,b]$ is the *total energy* of the solution on I (at time t):

$$E_I(t) = \int_a^b e(x,t)dx = \frac{1}{2}\int_a^b u_t^2 dx + \frac{c^2}{2}\int_a^b u_x^2 dx,$$

where the terms on the right may be thought of as"kinetic" and "potential" (elastic) energy over the interval.

These quantities have many interesting properties. For instance, the derivative of E with respect to t is:

$$\frac{dE_I}{dt} = \int_a^b (u_t u_{tt} + c^2 u_x u_{xt})dx = \int_a^b u_t(u_{tt} - c^2 u_{xx})dx + c^2 u_x u_t|_{x=a}^{x=b},$$

using integration by parts in the $u_x u_{xt}$ term. If u is a solution, the integral is zero, and we conclude:

$$\frac{dE_I}{dt} = p(a,t) - p(b,t),$$

the "momentum imbalance" at the endpoints. In particular, if u is the solution of the Dirichlet problem ($u = 0$ at the endpoints) or the Neumann problem ($u_x = 0$ at the endpoints) on an interval $[0, L]$, we conclude *the total energy is constant in* t (that is, the energy over the whole interval $[0, L]$).

To apply this further, it is useful to define the "energy-momentum vector field" on the (x, t) plane (defined by a function $u(x, t)$):

$$\vec{T}(x,t) = (p(x,t), e(x,t)) = (c^2 u_x u_t, \frac{1}{2}(u_t^2 + c^2 u_x^2)).$$

Computing the divergence of this vector field, we find (*check*):

$$\text{div } \vec{T} = \partial_x p + \partial_t e = u_t(u_{tt} - c^2 u_{xx}) = u_t W[u],$$

where $W[u] = u_{tt} - c^2 u_{xx}$ is the "differential operator" defining the WE. This suggests we will get information from applying the *divergence theorem* to suitable regions D in the (x, t) plane:

$$\int\int_D \text{div } \vec{T} dxdt = \oint_{\partial D} \vec{T} \cdot \vec{N} ds,$$

where $\vec{N}$ is the outward normal to D.

We'll need an observation regarding the flux of $\vec{T}$ across a characteristic line segment. These segments have the form:

$$x - ct = M, x \in [a, b] \text{ or } x + ct = M, x \in [a, b],$$

and may be parametrized (respectively) by:

$$\sigma_-(x) = (x, \frac{1}{c}(x - M)), x \in [a, b], \quad \sigma_+(x) = (x, \frac{1}{c}(M - x)), x \in [a, b].$$

The 'upward' normals in these parametrizations are:

$$N_- = (-\frac{1}{c}, 1), \quad N_+ = (\frac{1}{c}, 1).$$

Thus the flux integrals of $\vec{T}$ are:

$$\int_{\sigma_-} \vec{T} \cdot \vec{N} ds = \int_a^b (p, e) \cdot (-\frac{1}{c}, 1) dx, \quad \int_{\sigma_+} \vec{T} \cdot \vec{N} ds = \int_a^b (p, e) \cdot (\frac{1}{c}, 1) dx.$$

Now there is a surprising algebraic observation about the integrands. Note that:

$$(p, e) \cdot (-\frac{1}{c}, 1) = c u_x u_t + \frac{1}{2}(u_t^2 + c^2 u_x^2) = \frac{1}{2}(u_t + c u_x)^2, \quad (p, e) \cdot (\frac{1}{c}, 1) = \frac{1}{2}(u_t - c u_x)^2.$$

This implies, for the fluxes with respect to the upward normals:

$$\int_{\sigma_-} \vec{T}\cdot\vec{N}ds = \frac{1}{2}\int_a^b (u_t+cu_x)^2_{|\sigma_-(x)}dx, \quad \int_{\sigma_+} \vec{T}\cdot\vec{N}ds = \frac{1}{2}\int_a^b (u_t-cu_x)^2_{|\sigma_+(x)}dx,$$

both positive! (And note we don't have to assume u is a solution of the WE for this.)

As a first application, consider the non-homogeneous WE on a bounded interval $[0,L]$, with Dirichlet or Neumann boundary conditions:

$$u_{tt}-c^2u_{xx} = f(x,t), \quad u(x,0) = u_0(x), u_t(x,0) = u_1(x), u = 0 \text{ or } u_x = 0 \text{ on } \{0,L\}.$$

Let $x_0 \in (0,L)$, $t_0 > 0$ and let D be the backwards light cone from (x_0,t_0). There are two cases:

(i) $x_0 > ct_0$ and $x_0 < L - ct_0$, where $t_0 < L/c$. Then D is a triangle in $[0,L] \times R_+$, bounded by a segment $[x_0 - ct_0, x + 0 + ct_0]$ at $t = 0$ and characteristic segments σ_- and σ_+ converging at (x_0,t_0).

(ii) $t_0 > L/c$ and either $x_0 < ct_0$, or $x_0 > L - ct_0$ or both. Then D is a polygon with four or five sides: in addition to a segment $I_{x_0}(0)$ at $t = 0$ and two characteristic segments σ_-, σ_+ converging at (x_0,t_0), we have one or two vertical segments V_0, V_L along $x = 0$ or $x = L$ (respectively).

From the divergence theorem, we write the energy balance in all cases as:

$$\int\int_D u_tW[u]dxdt = -\int_{I_{x_0}(0)} edx+\frac{1}{2}\int_{\sigma_-}(u_t+cu_x)^2dx+\frac{1}{2}\int_{\sigma_+}(u_t-cu_x)^2dx+\int_{V_L}pdt-\int_{V_0}pdt.$$

In particular, if $W[u] = 0$ in D, we have:

$$E[u; I_{x_0}(0)] = \frac{1}{2}\int_{\sigma_-}(u_t + cu_x)^2dx + \frac{1}{2}\int_{\sigma_+}(u_t - cu_x)^2dx + \int_{V_L} pdt - \int_{V_0} pdt.$$

We may think of this expression as an "energy dissipation balance sheet": the energy inside the light cone at time zero dissipates along σ_- and σ_+, and as momentum through the right "wall" V_L, and some more momentum may be entering through the "left wall" V_0 (some of these segments may be missing, as described above.)

In particular for Dirichlet or Neumann boundary conditions the 'momentum' at V_0 or V_L is zero. This leads to a *uniqueness* statement: the solution at (x_0,t_0) is uniquely determined by the initial conditions and the value of the external force $f(x,t)$ inside the light cone D. For suppose we had two solutions, for values of f coinciding in D and the same initial conditions. Then their difference $v(x,t)$ would satisfy $W[v]=0$ in D and have zero initial data (and zero momenta at V_0, V_L). So the energy balance would simply say:

$$\int_{\sigma_-}(v_t+cv_x)^2dx+\int_{\sigma_+}(v_t-cv_x)^2dx=0,$$

and this implies:

$$v_t+cv_x\equiv 0 \text{ on } \sigma_-, \quad v_t-cv_x\equiv 0 \text{ on } \sigma_+.$$

Let's say σ_- is present and consider the first relation. σ_- runs (for increasing x) from $(x_0-ct_0,0)$ to (x_0,t_0) and we have, differentiating v along $\sigma_-(x)$ via the chain rule:

$$\frac{d}{dx}v(x,\frac{1}{c}(x-M))=v_x+\frac{1}{c}v_t\equiv 0, \quad (M=x_0-ct_0).$$

So we conclude $v(x_0,t_0)=v(x_0-ct_0,0)=0$, showing the solution is uniquely determined by the data inside D.

Exercise 6. Consider a different region D, a 'truncated backwards light cone' with vertex (x_0,t_0):

$$D=\{(x,t)|t_1\le t\le t_2, |x-x_0|<c(t-t_0)\}, \text{ where } 0\le t_1<t_2<t_0.$$

Sketch D, verify it is bounded by two horizontal segments and two characteristic segments σ_-, σ_+, and show (*using the divergence theorem, applied to the vector field $\vec{T}$ in this region D*) that the energy balance for a solution of the wave equation $W[u]=0$ is:

$$E[I_{x_0}(t_1)]=E[I_{x_0}(t_2)]+\frac{1}{2}\int_{\sigma_-}(u_t+cu_x)^2dx+\frac{1}{2}\int_{\sigma_+}(u_t-cu_x)^2dx.$$

Here $E[I_{x_0}(t)]=\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)}e(x,t)dx$, the total energy of the solution on interval with center x_0, radius $c(t_0-t)$ at time t (which is the section of D at time t).

9.Well-posed problems.

10. Solution by eigenfunction expansion.

There is a powerful method to solve *linear* time-dependent boundary-value problems, based on bringing in concepts from Linear Algebra. It applies to general second-order linear differential operators acting on the space variable x, for example of the form:

$$L[f] = -a(x)f_{xx}+q(x)f, a(x) > 0, a(x) \text{ and } q(x) \text{ piecewise continuous on } [0, L].$$

The boundary conditions are encoded by requiring the solution $u(\cdot, t)$ to lie, for each t, in a vector space V of (smooth) functions in $[0, L]$. The initial data u_0, u_1 are elements of V, and we think of the equation as 'evolving u_0 in time' within the space V (with initial velocity u_1, in the case of an equation of second order in time.)

The WE in an interval $[0, L]$ may be written in the form:

$$u_{tt} = -c^2L[u], \quad L[u] = -u_{xx}.$$

We think of L as a differential operator acting on a vector space of functions of x depending on the boundary conditions. For Dirichlet, Neumann and periodic boundary conditions we take, respectively:

$$V_D = \{f : [0, L] \to \mathbb{R} \text{ smooth}, \ f(0) = f(L) = 0\}.$$

$$V_N = \{f : [0, L] \to \mathbb{R} \text{ smooth}, \ f'(0) = f'(L) = 0\}.$$

$$V_{per} = \{f : [-L, L] \to \mathbb{R} \text{ smooth}, \ f(-L) = f(L), f'(-L) = f'(L)\}.$$

(From now on we take periodic BCs on the interval $[-L, L]$, corresponding to period $2L$; 'smooth' means having continuous derivatives of all orders.)

An *eigenfunction* of the operator L in the space V is a function $\varphi \in V$ such that:

$$L\varphi = \lambda\varphi,$$

where $\lambda \in \mathbb{R}$ is the 'eigenvalue'. We denote by $E_\lambda \subset V$ the subspace of eigenfunctions for eigenvalue λ. For instance, the functions

$$\varphi_n(x) = \sin(\frac{n\pi x}{L}) \in V_D, n = 1, 2, 3, \ldots$$

are eigenfunctions of $L[f] = -f''(x)$ in $[0, L]$ with Dirichlet boundary conditions, with eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$. The eigenspaces E_{λ_n} are one-dimensional.

For Neumann boundary conditions, the eigenfunctions/eigenvalues of L in V_N are:

$$\varphi_n(x) = \cos(\frac{n\pi x}{L}), n = 1, 2, \ldots, \lambda_n = \frac{n^2\pi^2}{L^2}; \quad \varphi_0(x) \equiv 1, \lambda_0 = 0.$$

For periodic boundary conditions (V_{per}) in $[-L, L]$ the eigenfunctions are:

$$\varphi_n(x) = \cos(\frac{n\pi x}{L}), \psi_n(x) = \sin(\frac{n\pi x}{L}), n = 1, 2, \ldots, \lambda_n = \frac{n^2\pi^2}{L^2},$$

in addition to the constants, with eigenvalue $\lambda_0 = 0$. Thus for periodic boundary conditions:

$$\mathrm{dim} E_{\lambda_n} = 2, n \neq 0; \quad \mathrm{dim} E_0 = 1.$$

If $\varphi(x) \in V$ is an eigenfunction of L with eigenvalue λ, we look for a solution of the WE of the form:

$$u(x,t) = f(t)\varphi(x).$$

(That is, we 'separate the time variable'). Since $u_{tt} = f''(t)\varphi(x)$, we find:

$$f''(t)\varphi(x) = -c^2 L[u] = -c^2 f(t)L[\varphi] = -c^2\lambda f(t)\varphi,$$

and this leads to the second-order ordinary differential equation for $f(t)$:

$$f''(t) = -c^2\lambda f(t),$$

which has the general solution (if $\lambda > 0$):

$$f(t) = A\cos(c\sqrt{\lambda}t) + B\sin(c\sqrt{\lambda}t), \quad A = f(0), B = f'(0)/c\sqrt{\lambda}.$$

If $\lambda = 0$, the general solution is linear: $f(t) = A + Bt$.

The corresponding solution of the WE satisfies the boundary conditions encoded in V, and the initial conditions:

$$u(x,0) = A\varphi(x), \quad u_t(x,0) = Bc\sqrt{\lambda}\varphi(x).$$

Reversing this reasoning, if we have $u_0 = \varphi \in E_\lambda$, the function:

$$u(x,t) = \cos(c\sqrt{\lambda}t)\varphi(x) \in V$$

solves the WE with the boundary conditions of V and initial conditions $u(x,0) = \varphi(x), u_t(x,0) = 0$. On the other hand, the function:

$$u(x,t) = \frac{1}{c\sqrt{\lambda}}\sin(c\sqrt{\lambda}t)\varphi(x) \in V$$

solves the WE with the boundary conditions of V and initial conditions $u(x,0) = 0, u_t(x,0) = \varphi(x)$.

And now we get to the main step. Since the WE is linear (and homogeneous, no force term for now) the above reasoning generalizes immediately to *finite* linear combinations of eigenfunctions. That is, consider two finite sets of eigenfunctions $\varphi_{\lambda_i} \in E_{\lambda_i}, i = 1, \dots, M$ and $\varphi_{\mu_j} \in E_{\mu_i}, j = 1, \dots, N$ in V and take the finite linear combinations (which are also in V):

$$u_0(x) = \sum_{i=1}^{M} A_i \varphi_{\lambda_i}(x), \quad u_1(x) = \sum_{j=1}^{N} B_j \varphi_{\mu_j}(x).$$

Then the function:

$$u(x,t) = \sum_{i=1}^{M} A_i \cos(c\sqrt{\lambda_i} t)\varphi_{\lambda_i}(x) + \sum_{j=1}^{N} (\frac{B_j}{c\sqrt{\mu_j}}) \sin(c\sqrt{\mu_j} t)\varphi_{\mu_j}(x)$$

solves the WE on $[0,L]$ with the V boundary conditions and with initial conditions $u(x,0) = u(x)$, $u_t(x,0) = u_1(x)$.

If all functions in V were *finite* linear combinations of eigenfunctions of L, this prescription would completely solve the problem.

Exercise 7. Consider the initial-value problem for the wave equation with $c = 1$, with Dirichlet boundary conditions on the interval $[0,\pi]$, with initial position and initial velocity given respectively by the functions $u_0(x)$, $u_1(x)$ below. Write down the solution $u(x,t)$.

$$u_0(x) = 3\sin x - 5\sin 3x + 7\sin 5x, \quad u_1(x) = 2\sin 2x - 6\sin 4x + 8\sin 6x.$$

Ideally we would like the vector space V to admit a *basis* consisting of eigenfunctions for the linear differential operator L. Since V is a function space, it is unreasonable to expect it has a finite basis, but we may look for an *infinite* basis (consisting of eigenfunctions.) This is the moment to recall a few important definitions and a major result from Linear Algebra.

An *inner product* in a vector space V assigns to each pair of vectors v, w in V a real number $\langle v, w\rangle$ in a bilinear way (linear with respect to linear combinations in each entry). We also require it to be *symmetric* ($\langle v, w\rangle = \langle w, v\rangle$) and *positive-definite*: $\langle v, v\rangle \geq 0$ for all $v \in V$, and is zero only if $v = 0$.

The *norm* of a vector $v \in V$ is $||v|| = \sqrt{\langle v, v\rangle}$.
(This definition abstracts the properties of the usual dot product in $\mathbb{R}^n$.)

Let $L : V \to V$ be a linear transformation, where V is a vector space with an inner product $\langle \cdot, \cdot \rangle$. We say L is *symmetric* if $\langle Lv, w\rangle = \langle v, Lw\rangle$, for all $v, w \in V$. The matrices of symmmetric linear transformations of $\mathbb{R}^n$ (for the usual dot product) with respect to the standard basis are exactly the symmetric $n \times n$ matrices.

Theorem. Any symmetric linear transformation L of a (finite-dimensional) inner-product space V may be orthogonally diagonalized, in the sense that there exists an orthonormal basis $\mathcal{B}$ of V consisting of eigenvectors of L:

$$\mathcal{B} = \{e_1, \dots e_n\}, ||e_i|| = 1, \langle e_i, e_j\rangle = 0 \text{ if } i \neq j, Le_i = \lambda_i e_i \text{ for some } \lambda_i \in \mathbb{R}.$$

An important standard fact is that eigenvectors of a symmetric linear transformation with different eigenvalues are necessarily orthogonal. Here is the one-line fundamental proof: if $v \in E_\lambda, w \in E_\mu$,

$$(\lambda - \mu)\langle v, w\rangle = \langle \lambda v, w\rangle - \langle v, \mu w\rangle = \langle Lv, w\rangle - \langle v, Lw\rangle = 0,$$

since L is symmetric. Hence if $\lambda \neq \mu$ we must have $\langle v, w\rangle = 0$.

To generalize these very useful properties to infinite-dimensional function spaces and linear differential operators, we first need an inner product. For the spaces V on an interval $[a, b]$ defined above, a useful definition turns out to be the 'L^2 inner product':

$$\langle f, g\rangle := \int_a^b f(x)g(x)dx.$$

The fact that the differential operator $L[f] = -f_{xx}$ is symmetric for this inner product on V is an immediate consequence of integration by parts! Integrating the identity:

$$[(f_x)g]_x - [fg_x]_x = f_{xx}g - fg_{xx}$$

over $[a, b]$, we find:

$$\langle L[f], g\rangle - \langle f, L[g]\rangle = -\int_a^b (f_{xx}g - fg_{xx})dx = (fg_x - f_x g)|_{x=a}^{x=b},$$

and the right-hand side vanishes in any of the three standard cases $V = V_D, V_N, V_{per}$ (*check this*).

Note in particular this shows that the eigenfunctions listed earlier for the spaces V_D, V_N, V_{per} are *orthogonal* with respect to the L^2 inner product on V (for different eigenvalues). In the case of V_{per} we need to check directly that:

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{n\pi x}{L}dx = 0,$$

since they have the same eigenvalue $n^2\pi^2/L^2$. This is left as an *exercise.* We also find by direct computation:

$$\int_0^L \sin^2\frac{n\pi x}{L}dx = \int_0^L \cos^2\frac{n\pi x}{L}dx = \frac{L}{2}.$$

Thus the set of functions:

$$\{e_n(x) = (\frac{2}{L})^{1/2}\sin\frac{n\pi x}{L}, n \geq 1\}$$

is an *orthonormal* set in V_D (pairwise orthogonal, with unit norm) , for the L^2 inner product. Likewise for the set in V_N:

$$\{e_n(x) = (\frac{2}{L})^{1/2}\cos\frac{n\pi x}{L}, n \geq 1, e_0(x) \equiv \frac{1}{\sqrt{L}}\}.$$

For V_{per}, the following is an orthonormal subset in $[-L, L]$:

$$\{f_n(x) = \frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}, e_n(x) = \frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L}, n \geq 1; e_0(x) \equiv \frac{1}{\sqrt{2L}}\}.$$

Returning to finite-dimensional Linear Algebra for a moment, suppose an inner-product space V has an orthonormal basis $\{e_1, \ldots, e_n\}$. Then the coefficients in the expression of an arbitrary vector $v \in V$ in this basis may be easily found:

$$v = \sum_{i=1}^{n} c_i e_i \Leftrightarrow c_i = \langle v, e_i\rangle, \text{ for } i = 1, \ldots n.$$

Now we make a leap and just suppose *everything works the same way for symmetric linear differential operators in infinite-dimensional function spaces with an inner product!* That is, we assume this is true, find the corresponding statements and then will have to prove them directly.

Take the operator $L[f] = -f_{xx}$ in the Dirichlet space V_D in $[0, L]$, with the L^2 inner product. We already have an orthornormal set $(e_n(x))_{n\geq 1}$in V_D of eigenfunctions of L. If this set were actually a *basis*, then an arbitrary function $f \in V_D$ would admit the expansion:

$$f(x) = \sum_{n=1}^{\infty} B_n e_n(x), \text{ where } B_n = \langle f, e_n \rangle \text{ for } n \geq 1,$$

or explicitly:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

(Note $b_n = (2/L)^{1/2} B_n$.)

Repeating this reasoning for the Neumann space V_N in $[0, L]$, we suppose any $f \in V_N$ admits the expansion:

$$f(x) = \sum_{n=0}^{\infty} A_n e_n(x), \text{ where } A_n = \langle f, e_n \rangle \text{ for } n \geq 0,$$

or explicitly:

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \text{ where } a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, n \geq 0.$$

(Note $a_n = (2/L)^{1/2} A_n$ for $n \geq 1$, while $a_0 = \frac{2}{\sqrt{L}} A_0$.)

For the periodic case V_{per} in $[-L, L]$, we have:

$$f(x) = \sum_{n=0}^{\infty} A_n e_n(x) + \sum_{n=1}^{\infty} B_n f_n(x), \text{ where } A_n = \langle f, e_n \rangle, B_n = \langle f, f_n \rangle,$$

or explicitly:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, n \geq 0; b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, n \geq 1.$$

Fourier series. The representations of functions in V_D, V_N, V_{per} as infinite series of trigonometric functions are known as *Fourier series.* For periodic functions on the real line, with period $2L$, we write:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \text{ (full Fourier series)},$$

where:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, n \geq 0; b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, n \geq 1$$

are the *full Fourier coefficients* of f.

In the special case where f is 2L-periodic and *odd* (so $f(0) = f(L) = 0$), the terms involving cosine (and the constant) vanish, and we have:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \text{ (Fourier sine series) }, b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, n \geq 1.$$

If f is 2L-periodic and *even* (so $f'(0) = f'(L) = 0$, the terms involving sine vanish and we have:

$$f(x) \sim \frac{a_0}{2} + \sum_{i=1}^{\infty} a_n \cos \frac{n\pi x}{L} \text{ (Fourier cosine series) }, a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, n \geq 0.$$

We're using the symbol $\sim$ above (instead of $=$) since we don't know at this point exactly in what sense the function equals the infinite series. What we have here is really an 'approximation scheme': the expectation is that if you add enough terms in the series, you'll get better and better approximations to the 'true solution'. The conditions under which this is true, and in what sense, will be made precise later. In particular, the following two questions will need to be answered.

(A) Consider a 2L-periodic function $f(x)$ on the real line. The Fourier coefficients $\{a_n, b_n\}$ of f are defined whenever f is piecewise continuous. Consider the partial sums:

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}.$$

Under what conditions do we have $||s_N - f|| \to 0$ as $N \to \infty$? It turns out that $f \in C^1(\mathbb{R})$ (i.e., f continuous, with f' continuous) is enough. For the higher derivatives, we have $||f^{(k)} - s_N^{(k)}|| \to 0$ if $f \in C^{k+1}$.

If we're using Fourier series to build (i.e. approximate) solutions of the wave equation, for each t we have an expansion:

$$u(x,t) \sim \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\frac{n\pi x}{L} + b_n(t) \sin\frac{n\pi x}{L},$$

where (informally) we have the full Fourier series:

$$u(x,0) = u_0(x) \sim \frac{a_0(0)}{2} + \sum_{n=1}^{\infty} a_n(0) \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n(0) \sin\frac{n\pi x}{L},$$

$$u_t(x,0) = u_1(x) \sim \frac{a_0'(0)}{2} + \sum_{n=1}^{\infty} a_n'(0) \cos\frac{n\pi x}{L} + b_n'(0) \sin\frac{n\pi x}{L}.$$

From the earlier discussion, we see that:

$$a_n(t) = a_n(0) \cos\frac{n\pi ct}{L} + \frac{a_n'(0)L}{n\pi c} \sin\frac{n\pi ct}{L} (n \geq 1), a_0(t) \equiv a_0(0)$$

$$b_n(t) = b_n(0) \cos\frac{n\pi ct}{L} + \frac{b_n'(0)L}{n\pi c} \sin\frac{n\pi ct}{L} (n \geq 1),$$

and this motivates the following basic question:

(B) Let sets of real numbers $\{A_n\}_{n\geq 0}, \{B_n\}_{n\geq 1}$ be given, and for each $N \geq 1$ form the 2L-periodic function:

$$s_N(x) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}.$$

Under which conditions on the coefficients A_n, B_n do the functions $s_N(x)$ converge to a continuous, periodic function f (and in what sense)? Under which conditions is the limit f of class C^k? If the coefficients depend on t: $A_n(t), B_n(t)$, so does the limit function $f(x,t)$. Under which conditions will it be continuous (or differentiable, of class C^k) also in t?

These are important (and well-understood) issues, and precise answers will be given in due course.