Math 435 spring 2011- Exam 1, 2/16/2011 No credit for answers without justification. Closed books, closed notes, calculators OK. Time given: 50 min.

1. For the wave equation $u_{tt} - u_{xx} = 0$, $(x,t) \in \mathbb{R} \times \mathbb{R}$, with initial data:

$$u(x,0) = \varphi(x) = \begin{cases} 1, |x| \leq 1, \\ 0, |x| > 1 \end{cases}, \quad u_t(x,0) \equiv 0.$$

(i) [12]Sketch the graph of $u(x, t_0)$ vs. x, for a fixed $t_0 > 2$.
(ii) [12]Sketch the lines in the (x,t) plane where $u(x,t)$ is not continuous. What are their slopes?

2.[25] Let $u(x,t)$ be the solution of the heat equation $u_t - u_{xx} = 0$ on the real line ($x \in \mathbb{R}, t > 0$), with initial condition $u(x,0) = \varphi(x)$ (where φ is bounded). Show that if φ is an even function of x, then so is u.
Hint: Use uniqueness, or the solution formula:

$$u(x,t) = \int_{-\infty}^{\infty} h(y,t)\varphi(x-y)dy, \quad h(y,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}}.$$

3. Consider the heat equation $u_t - u_{xx} = 0$ in the interval $x \in (0,1)$, with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition $\varphi(x) = 4x(1-x)$.
(i) [13] Explain why $0 < u(x,t) < 1$ for all $t > 0$ and $x \in (0,1)$.
(ii) [13] Use uniqueness to show that $u(x,t) = u(1-x,t)$. (*Hint:* Start by verifying that $v(x,t) = u(1-x,t)$ is also a solution.)

4. Let $f(x)$ be an odd function of $x \in \mathbb{R}$, periodic with period 2π and equal to $\frac{\pi}{2} - x$ in the interval $(0,\pi)$. Consider the Fourier sine series $f(x) \sim \sum_{n \geq 1} b_n \sin nx$ (there is no need to compute the coefficients b_n.)
(i) [9]Find the pointwise limit of this Fourier series, for each $x \in [-\pi, \pi]$.
(ii)[8] Does the series converge uniformly in $[-\pi, \pi]$? Justify.
(iii)[8] Explain what it means to say "the Fourier series converges to f in $L^2[-\pi,\pi]$". Is this true in this case? Justify.