MATH 431-SUMMER 2012-HOMEWORK SET
Qualitative analysis of autonomous 2D systems.

1. Phase-plane analysis. Analyze the following systems by linearization at the critical points, including the items below.

SYSTEMS
1. $x' = x + x^2, y' = x + y^2$.
2. $x' = x + y - y^2, y' = -x + y - 2y^2$
3. $x' = x - x^2 - 2xy, y' = y - y^2 - 2xy$
4. $x' = x(1 - y), y' = y(1 - x)$

OUTLINE:

(a) Find the critical points, and classify the linearized system at each critical point.

(b) If there are saddles, compute the stable/unstable eingenspaces at each saddle (these will be tangent to the stable/unstable separatrices.)

(c) Find the α and ω limits of each saddle separatrix (when they exist.)

(d) Identify other invariant sets: basins of attractors and sources, other open sets, possibly the coordinate axes.

(e) Identify finite (or half-finite) intervals of existence, when possible

(f) Include a MATLAB plot (with saddle separatrices and basins of attractors/sources highlighted) including at least two typical trajectories for each possible asymptotic behavior as $t \to \pm\infty$.

EXAMPLE (seen in class)

$$x' = x(3 - x - y), y' = y(x - y - 1).$$

(a) Critical points: O=(0,0) (saddle); P= (3,0) (saddle), Q=(0,-1) (unstable node), R=(2,1) (stable spiral)

(b) $E^s(O) = \{c(0,1)\}, E^u(O) = \{c(1,0)\}, \quad E^s(P) = \{c(1,0)\}, E^u(P) = \{c(3,-5)\}$

(c) One arc of $W^s(O)$ has Q as α-limit, the other diverges; one arc of $W^u(O)$ has P as ω-limit, the other diverges.

One arc of $W^s(P)$ has O as α-limit, the other diverges; one arc of $W^u(P)$ has R as ω-limit, the other diverges (it is the boundary of the basin of Q).

(d) The coordinate axes are invariant, since $x = 0$ implies $x' = 0$ and $y = 0$ implies $y' = 0$.

The open first quadrant is the basin of attraction $W^s(R)$ of the sink (attractor) R; the region in the open lower half-plane bounded by one arc of $W^u(P)$ is the basin $W^u(Q)$ of the source (repellor) Q.

(e) Solutions with IC in $W^u(Q)$ or in the unstable separatices of saddles are defined for all negative time, solutions with IC in $W^s(R)$ or the stable separatices of saddles are defined for all positive time. In particular, one arc of $W^u(P)$ corresponds to a solution defined for all $t \in \mathbb{R}$.

2. Gradient systems.

For each of the following functions $F(x, y)$, sketch the pahse portrait (along the lines of problem 1, with the help of Matlab) for the gradient system $v' = -\text{grad } F(v)$.

(a) $F(x, y) = y \sin x$.

(b) $F(x, y) = x^2 - y^2 - 2x + 4y + 5$

3. Lotka-Volterra competition. Sketch the phase portraits (along the lines of problem 1, with the help of Matlab) for three numerical examples of Lotka-Volterra systems, one in each of the three cases:

(i) weak competition: $\lambda_1 < \frac{k_1}{k_2} < \frac{1}{\lambda_2}$.

(ii) strong competition: $\lambda_1 > \frac{k_1}{k_2} > \frac{1}{\lambda_2}$.

(iii) intermediate: $\lambda_1 < \frac{k_1}{k_2} < \lambda_2$

4. Lienard's theorem.

Show that the second-order scalar equation $x'' + (x^6 - x^2)x' + x = 0$, $x = x(t)$, admits a non-constant periodic solution.

THEORY QUESTIONS. (Not due as homework.)

I: A gradient system cannot have non-constant periodic solutions.

II: If an autonomous system in the plane admits a conserved quantity $E(x, y)$, and E is not constant on any open subset of the plane, then the system cannot have a limit cycle.