MATH 431-FINAL EXAM, 12/8/2014, 10:15-12:15

Instructions. Closed book, closed notes, no electronics. *Full details for full credit.* Good luck!

1. *From HW9.* [20] Let $f(x) = -\frac{2x}{1+x^2}$. Consider the initial-value problem:

$$x' = f(x), \quad x = x(t), x(0) = 1.$$

(i) Find the first Picard iterates $x_1(t), x_2(t)$, given $x_0(t) \equiv 1$.

(ii) Show that f is bounded on R ($|f(x)| \leq M$) and globally Lipschitz (find an M and a Lipschitz constant K).

(iii) Sketch the phase line diagram for this autonomous equation, and explain why $x(t)$ is defined and in $[0, 1]$, for all $t \geq 0$.

(iv) Provide an estimate for the error $|x(t) - x_2(t)|$, valid for all $t \geq 0$.

2. *From Exam 3.* [10] Find Green's function $G_s(t)$ for the given self-adjoint operator L, in the interval $[1,2]$:

$$L[y] = (\frac{1}{t^6}y')' + \frac{12}{t^8}y, \quad y = y(t),$$

with Dirichlet boundary conditions $G_s(1) = G_s(2) = 0, s \in [1,2]$. *Hint:* Note that $L[y] = 0$ is of Cauchy-Euler type.

3. *From Exam 3.* [10] For the Sturm-Liouville eigenvalue problem:

$$y'' + (\lambda + q(t))y = 0 \text{ in } [a, b], y'(a) = y(a), y'(b) = -y(b),$$

with $q(t) < 0$, prove that all the eigenvalues λ are positive.

Hint: Multiply the equation by $y(t)$ and integrate over $[a, b]$, using integration by parts and the boundary conditions.

4. *Discussion problem; done in class. 12/2.* [10] Consider the second-order equation:

$$y'' + \frac{a(t)}{t^2} y = 0, \quad y = y(t), t \in [1, \infty).$$

(i) Show that if $a(t) \leq \lambda$ in $[1, \infty)$, where $0 < \lambda < 1/4$, all (nontrivial) solutions are non-oscillatory in $[1, \infty)$.

(ii) Show that if $a(t) \geq \lambda$ in $[1, \infty)$, where $\lambda > 1/4$, all (nontrivial) solutions are oscillatory in $[1, \infty)$

Hint: Comparison with the Cauchy-Euler type equation $z'' + \frac{\lambda}{t^2} z = 0, z = z(t)$ (find its general solution in each case.)

5. *From Exam 1.* [10] Consider the linear system:

$$x' = -8x + 10y, \quad y' = -4x + 4y \quad x(t), y(t).$$

The eigenvalues are $-1 \pm 2i$.

(i) Find a real-valued basis of solutions for the system.

(ii) Sketch the phase plane diagram for the system.

6. *From Exam 1.* [20] For the competing species system in the first quadrant:

$$x' = x(1 - x - y), \quad y' = y(2 - y - 3x), \quad x(t) \geq 0, y(t) \geq 0,$$

sketch the phase-plane diagram, including the following steps.

(i) Find the equilibria.

(ii) Linearize the system at each equilibrium, classifying it as stable/unstable/saddle and node/spiral.

(iii) Sketch the phase-plane diagram in the first quadrant, including representative trajectories for each possible asymptotic behavior (as $t \to \infty$) and saddle separatrices (if any).

(iv) List the possible ω-limit sets for solutions starting in the closed first quadrant ($x(0) \geq 0, y(0) \geq 0$).

7. *From Exam 2.* [20] Consider the *damped pendulum* described by the second-order equation:

$$x'' + ax' + b\sin x = 0, \quad x = x(t), a > 0, b > 0.$$

a) Write down an equivalent first-order system.

b) Find and classify (based on linearization) the equilibria of this system.

c) Prove that there are no periodic solutions.

d) Show that the total energy:

$$E(x, x') = \frac{1}{2}(x')^2 + b(1 - \cos x)$$

is a Liapunov function for the sytem (i.e., nonincreasing along solutions.)