MATH 431-Take-home test-Due Tuesday, 11/25/14.

Instructions. This is individual work; you may not obtain assistance from anyone. You may use the texts for this course, online handouts or your class notes, and no other sources. Include detailed answers for full credit. (10 pts. per item.)

1. Write in self-adjoint form:

$$y'' + \frac{2}{t+1}y' + \frac{\lambda}{(t+1)^2}y = 0, \quad y = y(t), t \in (-1, \infty).$$

2. Find a homogeneous *self-adjoint* second-order linear differential equation that has the function given below as a solution:

$$x(t) = \sin(\frac{1}{t^5}), \quad t \in (0, \infty).$$

3. (i) Find Green's function for the given operator L, in the interval $[1, 2]$:

$$L[y] = t^2y'' - 6ty' + 12y.$$

(ii) Find the solution to the non-homogeneous boundary-value problem:

$$L[y] = 7t^5, \quad y(1) = 0, y(2) = 56.$$

4. (i) Find all eigenpairs for the periodic Sturm-Liouville problem:

$$y'' + \lambda y = 0, \quad y = y(t), y(-5) = y(5), y'(-5) = y'(5).$$

(ii) Show that eigenfunctions with different eigenvalues (and the periodic boundary conditions in (i)) are orthogonal in the interval $[-5, 5]$. (Indicate explicitly how the boundary conditions are used.)

5. Show that the differential equation:

$$(\frac{y'}{\ln t})' + \frac{3}{\sqrt{t}}y = 0, \quad y = y(t), t \geq 2,$$

is oscillatory in $[2, \infty)$. *Hint:* Fite-Wintner theorem (see the online handout, or [KP]). Explain in detail how its hypotheses apply here.

6. For the Sturm-Liouville eigenvalue problem:

$$y'' + (\lambda + q(t))y = 0 \text{ in } [a, b], \alpha y(a) + \beta y'(a) = 0, \gamma y(b) + \delta y'(b) = 0,$$

prove that all the eigenvalues are positive if $q(t) < 0$, $\alpha\beta < 0, \gamma\delta > 0$.

Hint: Multiply the equation by $y(t)$ and integrate over $[a, b]$, using the boundary condition.

7. Show that any solution of *Airy's equation*

$$y'' + ty = 0, \quad y = y(t),$$

is oscillatory in $(0, \infty)$, but has at most one zero in $(-\infty, 0)$.

Hint: Sturm comparison. Consider separately intervals of the form $[a, b]$ and $[-b, -a]$, with $a > 0$ and $b > 0$ arbitrarily large.

8. Show that between two consecutive zeros of any nontrivial solution of the equation:

$$(p(t)y')' + q(t)y = 0, \quad p(t) > 0, q(t) > 0, y = y(t)$$

there exists *exactly one* maximum or minimum point.

Hint: We may assume $y(t) > 0$ between the zeros. First show there must be a maximum point, then that there can't be more than one. (Can there be a *minimum* point between the zeros in this case? Expand the equation.)