MATH 431-Take-home test-Due Tuesday, 11/4/14.

Instructions. This is individual work; you may not obtain assistance from anyone. You may use the texts for this course, online handouts or your class notes, and no other sources. Include detailed answers for full credit.

1.[15] Find conserved quantities for the linear vector fields:

$$a)\mathbf{F}(x,y) = (-3x-2y, 5x+3y) \quad \text{(a center);}$$

$$b)\mathbf{F}(x,y) = (-5x-3y, 6x+4y) \quad \text{(a saddle).}$$

Hint: Try quadratic functions, $Ax^2 + 2Bxy + Cy^2$.

c) Can linear nodes (stable or unstable) or spirals (stable or unstable) admit non-constant conserved quantities? If not, why not?

2. [10] Let $\mathbf{F} = -\nabla V$ be a gradient vector field in R^2 (You may assume the critical points of V are isolated). Show that if $p \in \mathbb{R}^2$ and the orbit $\mathcal{O}^+(p)$ is contained in a compact set, then the limit set $\omega(p)$ must be an equilibrium. (Notice *be*–a single equilibrium–as opposed to *contain*.)

Hint: Recall that gradient vector fields don't have (non-constant) periodic orbits. Use the fact that V is decreasing along solutions.

3.[20] Consider the autonomous system in R^2:

$$y_1' = -y_2 + y_1(y_1^2+y_2^2)\sin(\frac{\pi}{\sqrt{y_1^2+y_2^2}}), \quad y_2' = y_1 + y_2(y_1^2+y_2^2)\sin(\frac{\pi}{\sqrt{y_1^2+y_2^2}}).$$

a) Find the equivalent system in polar coordinates (r,θ).

b) Sketch the phase-line diagram for the r equation, and

c) The r vs. t graph.

d) Answer: what are all the limit sets $\omega(p)$, for each $p \in R^2$?

4.[10] Show that the system in the plane:

$$x' = 2x - x^5 - y^4 x, \quad y' = y - y^3 - yx^2$$

has no periodic solutions.

Hint: show that all the equilibria are on the coordinate axes, and that the axes are invariant. Sketch the phase plane diagram and explain why a closed orbit would lead to a contradiction.

5.[15] a) Find and classify (based on linearization) the equilibria of the system:

$$x' = y, y' = -b\sin x - ay, \quad a,b > 0.$$

b) Sketch the phase plane diagram.

c) Prove that it does not have periodic solutions.

6. [10] a) Which linear systems in R^2 are Hamiltonian? Justify.

b) Give an example of a Hamiltonian system in R^4 (show that it is Hamiltonian.)

7. [10]A vector field $\mathbf{F}$ in the plane has a periodic orbit γ.

a) Can $\mathbf{F}$ have exactly two hyperbolic equilibria in the interior of γ? If so, what are their possible types? (Saddle/node or spiral). Justify.

b) Can it have exactly three hyperbolic equilibria enclsed by γ? What are their possible types? Sketch examples.

(Recall *hyperbolic* means no eigenvalue of the linearization has zero real part.)

8.[10] A linear system in R^3 has eigenvalues $2 \pm i$ and -3, with eigenvectors (respectively): $(3 \pm 2i, 1 \pm i, -1), (-1, 2, 1)$. Find a basis of solutions for the system (real-valued).