SOME EXAMPLES OF AUTONOMOUS SYSTEMS IN THE PLANE.

**Example 1.** *Simple predator-prey system.*

*1. The system.* Consider the system in the plane:

$$x' = x(1 - \frac{x}{k}) - axy, \quad y' = bxy - y, \quad x(t) \geq 0, y(t) \geq 0.$$

$x(t)$ is the prey population, $y(t)$ the predator population. In the absence of the predator species, the prey population is modeled by logistic growth with carrying capacity $k$; in the absence of prey, the predator population decays exponentially to zero. $a, b > 0$ are interaction coefficients.

*2. Initial sketch of the phase plane diagram.* The axes are invariant. The other nullclines are the lines $\frac{x}{k} + ay = 1$ and $x = \frac{1}{b}$. Thus, there are two cases: (i) $bk < 1$. Then the nullclines don't intersect in the first quadrant. (ii) $bk > 1$; then there is an intersection. *In each case, sketch the nullclines and the vector field on them. Then sketch the vector field at one point in each region bounded by the nullclines.*

*3.Equilibria and linearization.* In *case (i)* there are two equilibria, both on the axes: the origin and $(k, 0)$. In case (ii) there is a third equilibrium at $(\frac{1}{b}, \frac{1}{a}(1 - \frac{1}{kb}))$. The differential (Jacobian matrix) of the vector field $\mathbf{F}$ is the $2 \times 2$ matrix (by columns):

$$D\mathbf{F}(x, y) = [(1 - \frac{2x}{k} - ay, by)|(-ax, bx - 1)].$$

*Establish the following:* The origin is a saddle point. The equilibrium $(k, 0)$ is a stable node in case (i), a saddle in case (ii). The third equilibrium (in case (ii)) is a stable node or stable spiral.

Thus, in each case the equilibria are *hyperbolic* (the eigenvalues have nonzero real part.) This implies that, locally near each equilibrium, the phase plane picture looks like that of the linearization. *Use this fact to complete a possible sketch of typical trajectories in the phase plane, in both cases.*

*4. Invariant regions.* The goal is to find compact forward-invariant regions for the system containing all the equilibria. In such a region solutions are guaranteed to exist for all $t \geq 0$, and their asymptotic behavior can be analyzed.

In *case (i)* this is easy. On the line $\{x = \frac{1}{b}\}$ we have $x' < 0$; so the vector field $\mathbf{F}$ points into the interior of the strip $\{0 \leq x \leq 1/b, y \geq 0\}$

on this boundary component (and is tangent to the boundary on the axes). Thus, this strip is invariant (but not compact). Moreover, on any horizontal segment of the form $\{0 \leq x \leq \frac{1}{b}, y \equiv y_0\}$ one has $y' \leq 0$. This implies any rectangle:

$$R = \{0 \leq x \leq \frac{1}{b}, 0 \leq y \leq y_0\}, \quad (y_0 > 0)$$

is a compact invariant region including all the equilibria in its interior. *Verify these claims. Sketch such a rectangle, including a plot of the vector field at its boundary points.*

From this we can describe the asymptotic behavior of any solution with initial condition in the interior of the strip: it converges, as $t \to +\infty$, to the equilibrium $(k, 0)$. (The predator population becomes extinct, the prey population reaches its carrying capacity.)

*Case (ii)* takes a little more work to analyze. The vector field at points $(k, y_0)$ (on the line $x = k$) takes the value:

$$\mathbf{F}(k, y_0) = (-aky_0, y_0(bk - 1)).$$

This has negative first component, positive second component. Thus the strip $\{x = k, y \geq 0\}$ is invariant (but not compact). And this time trajectories starting on the line $\{x = \frac{1}{k}\}$ move upward for a while, until they reach the line $x = \frac{1}{b}$ and start moving down, appearing to spiral towards the equilibrium (*sketch this behavior*). (In particular, this happens to the trajectory that for $t \to -\infty$ converges to the saddle point at $(k, 0)$.) So the rectangular regions we used in case (i) are not invariant, and we need to find a different one.

To find a compact invariant region containing all the equilibiria, consider trapezoidal regions bounded by the horizontal segments:

$$\{0 \leq x \leq k, y = 0\}, \quad \{0 \leq x \leq \frac{1}{b}, y = C\},$$

the vertical segments:

$$\{x = 0, 0 \leq y \leq C\}, \quad \{x = k, 0 \leq y \leq y_0\}$$

and the line segment:

$$\alpha(\frac{x}{k} - 1) + (y - y_0) = 0, \quad \frac{1}{b} \leq x \leq k, y_0 \leq y \leq C.$$

Here the constants $\alpha > 0$ and $y_0 > 0$ are to be chosen, and $C > 0$ is determined by them:

$$C = y_0 + \alpha(1 - \frac{1}{kb}).$$

*Sketch this region.* It is easy to see that, on the vertical and horizontal boundary segments, the vector field $\mathbf{F}$ points into the region, or is tangent to the boundary (*check this*). This will also be true on the remaining boundary segment, if we can show $\mathbf{F} \cdot n \leq 0$ there, where $n = (\alpha, 1)$ is the outward normal to the segment. An elementary calculation (*check*) yields:

$$\mathbf{F} \cdot n = \alpha(1 - \frac{x}{k})(\frac{x}{k} - a + bx - 1) + y_0(1 - a\alpha\frac{x}{k}).$$

The second term is bounded above (in the range $1/b \leq x \leq k$) by $y_0(1 - a\alpha/kb)$, and will be negative if we choose $\alpha > kb/a$. Fix such an $\alpha$. Then the first term (in the range $1/b \leq x \leq k$) will be bounded by $\alpha(1 - \frac{1}{kb})|bk - a|$, as the reader can easily check. So all we need to do is, after fixing $\alpha$, choose $y_0 > 0$ large enough (depending on $a, b, k$ and $\alpha$) so that:

$$\alpha(1 - \frac{1}{kb})|bk - a| + y_0(1 - \frac{a\alpha}{bk}) < 0.$$

Then $\mathbf{F} \cdot n$ will be negative on this line segment.

**Example 2.** *Predator-prey with logistic growth for predator.* A different model allows for survival of the predator species in the absence of prey. Consider the system:

$$x' = x(k_1 - x - c_1 y), \quad y' = y(k_2 - y + c_2 x), \quad x(t) \geq 0, y(t) \geq 0.$$

Here $k_1, k_2$ are carrying capacities and $c_1, c_2$ are interaction coefficients.

*Nullclines and equilibria.* The axes are invariant, with the phase line diagrams for logistic growth: equilibria at $(k_1, 0)$, $(0, k_2)$ and $(0, 0)$. The nulclines are the lines $x + c_1 y = k_1$ and $y = c_2 x + k_2$. There are two cases to consider: (i) $k_1 > c_1 k_2$: the nullclines intersect in the first quadrant, and there is a fourth equilibrium $(a, b)$ with $a > 0, b > 0$. (ii) $k_1 < c_1 k_2$: the nullclines don't intersect in the first quadrant, and there are no other equilbiria. *Sketch the phase diagram in both cases, including the equilibria and the direction of the vector field on the nullclines and in each region they bound.*

*Linearization.* Compute the matrix $D\mathbf{F}$ at each equilibrium point, and establish the following. The origin is an unstable node and $(0, k_1)$ a saddle, in both cases. The equilibrium $(0, k_2)$ is a saddle point in case (i), a stable node in case (ii). In case (i) the equilibrium $(a, b)$ is a stable node or stable spiral.

*Use the information from linearization to complete the phase diagram, sketching a few sample trajectories in each case. Sketch compact invariant regions containing all the equilibria.*

**Example 3.** *Competing species (Lotka-Volterra).* (See [Waltman], 2.8.) Consider the system for two populations:

$$x' = x(k_1 - x - c_1 y), \quad y' = y(k_2 - y - c_2 y), \quad x(t) \geq 0, y(t) \geq 0.$$

Here $k_1, k_2 > 0$ are carrying capacities and $c_1, c_2$ are interaction coefficients.

*Nullclines and equilibria.* The axes are invariant, with the logistic growth phase line picture. The origin and the points $(k_1, 0), (0, k_2)$ are equilibria. The nullclines have equations $x + c_1 y = k_1, c_2 x + y = k + 2$. There are *FOUR* cases to consider:

(1A)$k_1/c_1 > k_2$ and $k_2/c_2 < k_1$;

(1B) $k_1/c_1 < k_2$ and $k_2/c_2 > k_1$;

(2) $k_2 < k_1/c_1$ and $k_1 < k_2/c_2$;

(3) $k_2 > k_1/c_1$ and $k_1 > k_2/c_2$.

In cases (1A,B) there are no other equilibria. In cases (2),(3) there is a fourth equilibrium $(a, b)$ in the open first quadrant. *For each case: sketch the nullclines and representative plots of the vector field on them, and in each of the regions they define.*

*Linearization. Compute the linearization $D\mathbf{F}$, and establish the following.* (i) The origin is always an unstable source. (ii) In case (1A), the point $(k_1, 0)$ is a stable node, while $(0, k_2)$ is a saddle. In case (1B), this classification is reversed. (In these cases, every solution starting in the open first quadrant converges to the stable node: one of the species becomes extinct–that with the largest interaction constant.)

(iii) Taking the product of the conditions defining case (2), we see that $c_1 c_2 < 1$. This is the "weakly interacting" case. In this case, show that the equilibrium $(a, b)$ in the open first quadrant is a stable node, and both $(k_1, 0)$ and $(0, k_2$ are saddles. Any solution starting in the open first quadrant converges to $(a, b)$ (asymptotically stable coexistence.)

(iv) In case (3), we find $c_1 c_2 > 1$. (This is the "strongly interacting" case.) Then $(a, b)$ is a saddle point, and both $(k_1, 0)$ and $(0, k_2)$ are stable nodes. There is no stable coexistence, and one of the species becomes extinct eventually. Which one becomes extinct depends on which side of the unbounded stable saddle separatrix $W^s(a,b)$ the initial condition $(x_0, y_0)$ is (hence the name *separatrix*–it separates two basins of attraction). Thus, the asymptotic state *depends sensitively* on the initial condition.

*Complete the analysis by sketching a few sample trajectories in each case, illustrating the possible behaviors of solutions as* $t \to +\infty$. Sketch a compact invariant set in each case.

**Example 4.** *The frictionless pendulum.* The second-order equation of motion is:

$$x'' + \sin x = 0,$$

or as a first-order hamiltonian system:

$$x' = y, \quad y' = -\sin x,$$

with conserved quantity (or hamiltonian):

$$E(x,y) = \frac{y^2}{2} + 1 - \cos x.$$

The equilibria of the system are the points $(n\pi, 0), n \in \mathbb{Z}$.

*Exercise.* These are also the critical points of $E(x,y)$. By the second-derivative (Hessian) test, the points $(n\pi, 0)$ with $n$ even are local minima, while $(n\pi, 0)$ with $n$ odd are saddle points. Indeed, $E$ attains its global minimum value (namely, 0) at $(n\pi, 0)$ with $n$ even.

*Exercise.* Linearize the vector field $\mathbf{F}(x,y) = (y, -\sin x)$ (that is, compute its differential $D\mathbf{F}(x,y)$, a $2 \times 2$ matrix). The linearized system is a saddle at $(n\pi, 0)$ with $n$ odd, a center when $n$ is even. Thus the *nonlinear* system defined by $\mathbf{F}$ has a saddle-type phase diagram near $(n\pi, 0)$ with $n$ odd; but when $n$ is even, we can't conclude stability (since the eigenvalues have zero real part).

To describe the orbits of the non-linear system near $(n\pi, 0)$ with $n$ even, recall the orbits are contained in level sets of $E$, and *are* connected components of level sets of $E$ when these contain no critical points. The points $(n\pi, 0)$ with $n$ even are minima of $E$, so the connected components of level sets near these critical points are closed curves (corresponding to periodic solutions of the system). We say *connected components* because, since $E$ is $2\pi$-periodic in $x$, each of these closed curves admits infinitely many disjoint copies (shifted to the left/right by $2\pi$ along the $x$-axis), where $E$ takes the same value.

In fact, one can be more precise: each orbit corresponding to a value of $E$ in the range $0 < E < 2$ is periodic. (Note that $E = 2$ is the energy value at the equilibria $(n\pi, 0)$ with $n$ odd, which are critical points of saddle type for $E$). To see this, consider the level set of $E$ (in the closed strip $|x| \leq \pi$):

$$L_E = \{(x, y); |x| \leq \pi, \frac{y^2}{2} + (1 - \cos x) = E\}.$$

If $0 < E < 2$, we have $-1 < E - 1 < 1$, hence may write $E - 1 = -\cos x_0$, for some $x_0$ with $0 < x_0 < \pi$. Then for $(x, y) \in L_E$:

$$0 \leq \frac{y^2}{2} = \cos x - \cos x_0,$$

which implies $|x| \leq x_0 < \pi$. So $L_E$ is contained in a rectangle

$$\{|x| \leq x_0 < \pi, |y| \leq \sqrt{2E}\},$$

bounded away from the boundary $\pm\pi$ of the strip and without critical points of $E$. Thus this component of a level set must be a closed curve.

For $E = 2$, the orbits are saddle connections, going from one saddle at $(-\pi, 0)$ to another at $(\pi, 0)$, for increasing or for decreasing time (there are two connections in the closed strip $\{|x| \leq \pi\}$).

For $E > 2$, the level set consists of two open curves (one on the half-plane $y > 0$, the other with $y < 0$). To see this, it is enough to show that, on the closed strip $|x| \leq \pi$, such a level set has points of the form $(\pm\pi, y_0)$. Indeed, since $E > 2$, we can let $y_0 = \sqrt{2(E - 2)}$; then it is easy to see that $(\pm\pi, y_0) \in L_E$.

*Exercise:* Given this information, sketch the phase plane diagram for the pendulum in the strip $\{|x| \leq 2\pi, y \in R\}$, including the saddle connections and two orbits in each of the ranges $E < 2$, $E > 2$.

**Example 5.** *Pendulum with friction.* The second-order equation of motion is:

$$x'' + rx' + \sin x = 0, \quad r > 0.$$

The equivalent first-order system:

$$x' = y, \quad y' = -\sin x - ry$$

is no longer Hamiltonian.

*Linearization.* The equilibria are still $(n\pi, 0)$, $n \in \mathbb{Z}$. Computing the linearization, one finds they are saddles for $n$ odd, stable nodes or stable spirals for $n$ even, depnding on whether $r > 2$ or $r < 2$ (*verify this.*).

One can use the Liapunov-La Salle invariance theorem to estimate the basin of attraction of the origin. First, *verify* that the energy for the undamped pendulum is a Liapunov function for the damped pendulum:

$$V(x,y) = \frac{y^2}{2} + 1 - \cos x \Rightarrow \dot{V} = -ry^2 \leq 0 \text{ in } R^2.$$

Thus $\dot{V} = 0$ only when $y = 0$. To apply the theorem, let $U$ be the bounded open set $U = (-\pi, \pi \times (-2, 2)$. Then if we fix an $E < 2$, the sublevel set

$$S_E = \{(x,y); |x| < \pi, V(x,y) \leq E\}$$

is contained in $U$, and the only invariant set in $Z = \{(x,y) \in S_E; \dot{V} = 0\}$ is the origin itself. Thus by the Lasalle-Liapunov invariance theorem, the origin is asymptotically stable, and its basin of attraction contains $S_E$.

(For other examples of application of the theorem, see [Waltman], pp. 138-140.)

**Example 6.** *When linearization fails.* (see [Waltman], p. 129.) Consider the system:

$$x' = -y + x(x^2 + y^2), \quad y' = x + y(x^2 + y^2).$$

Linearizing at the origin, we obtain a *center.* So we can't conclude anything about the solutions of the nonlinear system. In polar coordinates, we obtain*verify*:

$$r' = r^3, \quad \theta' = 1.$$

Thus the origin is the only equilibrium, and the nonlinear system has an unstable spiral. In fact, *solutions spiral away to infinity in finite time.* To see this, note that the solution starting at $(r_0, \theta_0)$ (in polar coordinates) is:

$$r(t) = \frac{r_0}{\sqrt{1 - 2r_0^2 t}}, \quad \theta = \theta_0 + t,$$

and $r(t)$ is defined only for $0 \leq t \leq \frac{1}{2r_0^2}$.