STABILITY OF EQUILIBRIA AND LIAPUNOV FUNCTIONS.

By *topological properties* in general we mean qualitative geometric properties (of subsets of R^n or of functions in R^n), that is, those that don't depend on measuring "distance" or 'angle". Examples: continuity of functions, convergence of sequences, compactness of sets (see below.)

1. Topological definitions. The open ball with center p and radius $r > 0$ in R^n is the set:

$$B_r(p) = \{x \in R^n; ||x - p|| < r\}.$$

A set $U \subset R^n$ is *open* if whenever $p \in U$ the balls $B_r(p)$ are contained in U, for all sufficiently small $r > 0$.

A set $F \subset R^n$ is *closed* if it contains all its limit points:

$$(x_k) \text{ sequence in } F, \lim x_k = x \Rightarrow x \in F.$$

If $F : R^n \to R$ is a function, *level sets* and *sublevel sets* of F are any sets of the form (respectively);

$$L_c = \{x; F(x) = c\}; \quad S_c = \{x; F(x) \leq c\}, \text{ where } c \in R.$$

If F is continuous, level sets and sublevel sets of F are closed. (Depending on the value of c, they may be empty!)

A set $K \in R^n$ is *compact* if it is closed and bounded. (*Bounded* means it is contained in some ball $B_R(0)$, where R is sufficiently large. Compact sets K have an important property: any sequence of points in K has a convergent *subsequence.*

A continuous function $F : R^n \to R$ is *proper* if either $\lim_{||x|| \to \infty} F(x) = +\infty$ or $\lim_{||x|| \to \infty} F(x) = -\infty$. The first statement is equivalent to:

$$(\forall M > 0)(\exists R > 0)(\forall x)(||x|| > R \Rightarrow f(x) > M).$$

For example, any polynomial function of the coordinates (x_i) in which they all appear with even degree, and positive coefficients, is a proper function. Say, in R^3:

$$F(x_1, x_2, x_3) = 2x_1^2 + 4x_2^4 x_3^6 + 8x_2^8.$$

Level sets and sublevel sets of continuous proper functions are compact.

A continuous function F on R^n is of class C^1 if all its first-order partial derivatives $\frac{\partial F}{\partial x_i}$ are themselves continuous functions on R^n. More generally, we say F is of class C^k if all its partial derivatives of total order up to k are continuous functions.

The level sets L_c of C^1 functions are C^1 $(n-1)$-dimensional surfaces (that is, have well-defined tangent planes of dimension $n-1$, varying continuously along the surface) without self-intersections, *except* at critical points x of L_c (that is, $F(x) = c, \nabla F(x) = 0$). Recall from Calculus that at a regular point of L_c (that is, a non-critical point) the normal direction is given by the gradient vector ∇F.

Combining all the above, we see that if F is C^1 and proper, those level sets of F which do not contain critical points are bounded C^1 surfaces (of dimension $n-1$) without self-intersections, but possibly consisting of more than one piece. The corresponding sublevel sets are compact subsets of R^n, bounded by C^1 surfaces.

2. Definitions for vector fields. A vector field $\mathbf{F} : R^n \to R^n$ is of class C^k if each if its component functions is. By the existence-uniqueness theorem, if $\mathbf{F}$ is a C^1 vector field, for each $x \in R^n$ there exists a unique solution $x(t) = \phi(t, x)$ to the initial-value problem $x' = \mathbf{F}(x), x(0) = x$, defined on a maximal interval: $t \in I_x = (\alpha_x, \omega_x)$ (where α_x or ω_x may be infinite.) We have the *semigroup property*, which follows fom uniqueness [KP]:

$$\phi(t, (\phi(s, x))) = \phi(t + s, x) = \phi(s, \phi(t, x))), \quad \text{if } t, s, s + t \in I_x.$$

The function $\phi(t, x)$ from $D = \{(t, x); x \in R^n; t \in I_x\}$ to R^n is referred to as the *flow* of X, and the alternative notation $\phi_t(x) = \phi(t, x)$ is often used.

In the following we're mostly interested in the case when ω_x is infinite, or when $I_x = R$. It follows from the E/U theorem that the first is guaranteed if the *forward orbit* is contained in a bounded set:

$$\mathcal{O}^+(x) = \{\phi(t, x); t \in [0, \omega_x)\}.$$

The full orbit is: $\mathcal{O}(x) = \{\phi(t, x); t \in I_x\}$. if we know the full orbit is contained in a bounded set, it follows that $I_x = R$.

The main interest is in the asymptotic behavior of solutions for large time: is it periodic? Does it converge to an equilibrium state? Does it exhibit oscilations for arbitrarily large time? Does it come arbitrarily close to different equilibrium states, for arbitrarily large times?

These 'states approximated by the solution' for large times are captured by the notion of ω-limit set. Suppose the solution is defined for all positive times, so $\omega_x = +\infty$ Then we defined the *ω-limit* set of x as:

$$\omega(x) = \{y \in R^n; \exists (t_k)_{k\geq 1} \text{ in } R^n; t_k \to +\infty \text{ and } \phi(t_k, x) \to y\}.$$

That is, $\omega(x)$ is the set of all limit points of the forward orbit $\mathcal{O}^+(x)$. The *α-limit set* $\alpha(x)$ is denfined analogously, when $\alpha_x = -\infty$ (let $t_k \to -\infty$).

When non-empty, $\omega(x)$ and $\alpha(x)$ are closed in R^n. For instance, if (x_k) is a sequence in $\omega(x)$ we may find for each $k \geq 1$ sequences $(t_n^k)_{n\geq 1}$ tending to $+\infty$ with n, so that $\lim_n \phi(t_k^n, x) = x_k$. Then if $x_k \to y \in R^n$, considering the triangle inequality:

$$||\phi(t_n^k, y) - x|| \leq ||\phi(t_n^k, x) - x_k|| + ||x_k - y||$$

shows that if we take k large enough so that x_k is close to y, and then $n(k)$ large enough so that $\phi(t^{n(k)}, x)$ is close to x_k, then for the sequence $s_k = t_k^{n(k)}$ we have $\phi(s_k, x)$ is close to y and $s_k \to +\infty$, so $y \in \omega(x)$.

The sets $\omega(x)$ or $\alpha(x)$ may be empty. One condition that guarantees $\omega(x) \neq \emptyset$ is: $\mathcal{O}^+(x)$ is a bounded set. If K is compact and $\mathcal{O}^+(x) \subset K$, then $\omega(x)$ is a non-empty subset of K. $\omega(x)$ has exactly one point $\bar{x}$ if and only if $\lim_{t\to+\infty} \phi(t, x) = \bar{x}$.

A set $A \subset R^n$ is *invariant* under the flow of a vector field $\mathbf{F}$ if $x \in A \Rightarrow \phi(t, x) \in A, \forall t \in I_x$; *forward-invariant* or *backward-invariant* if we take $t \in [0, \omega_x)$ or $t \in (\alpha_x, 0]$.

$\omega-$limit sets are forward invariant: if $y \in \omega(x)$, by definition we may find a sequence $t_k \to +\infty$ so that $\phi(t_k, x) \to y$. Given $t > 0$, the semigroup property gives:

$$\phi(t + t_k, x) = \phi_t(\phi(t_k, x)) \to \phi_t(y),$$

using the fact that $\phi_t(x)$ depends continuously on x. Since $t + t_k \to +\infty$, this shows $\phi_t(y) \in \omega(x)$. (Analogously, α-limit sets are backwards invariant under the flow of $\mathbf{F}$.)

3. Liapunov functions. Recall that a function $E : R^n \to R$ is a *conserved quantity* for the vector field $\mathbf{F}$ if, for every $x \in R^n$, the function $E(\phi(t, x))$ is constant in t (briefly: E is constant along solutions). By the chain rule, this happens exactly when:

$$\dot{E}(x) := \nabla E(x) \cdot \mathbf{F}(x) = 0, \quad \forall x \in R^n.$$

When this happens, all orbits of $\mathbf{F}$ are contained in level sets of E. (Recall that these are $(n-1)$-dimensional surfaces, at points where $\nabla E \neq 0$.) If

$n = 2$ the level sets of E are curves, so the orbits of $\mathbf{F}$ can be identified as arcs of level sets (often beginning or ending at a singular point of E.)

Systems with conserved quantities are relatively rare, other than in conservative Classical Mechanics. But it is also very useful to find functions that decrease (or, at least, do not increase) along solutions.

Definition. A function $V : R^n \to R$ is a *Liapunov function* for the vector field $\mathbf{F}$ in the open set $U \subset R^n$ if $\nabla V \cdot \mathbf{F} \leq 0$ holds in U; a *strict* Liapunov function if we have $\nabla V \cdot \mathbf{F} < 0$ in U.

Geometrically, for a strict Liapunov function, at any boundary point of a sublevel set S_c, the vector field $\mathbf{F}$ points 'into' the sublevel set, since $\mathbf{F}$ makes an obtuse angle with the outward normal ∇V. (If the Liapunov function is not strict, $\mathbf{F}$ may be tangent to level sets at some points.)

Thus V is non-increasing along orbits $\phi(t, x)$ as long as they are in U (or decreasing in the strict case.) The first use of Liapunov functions is in finding invariant sets:

Proposition 1. Suppose $V : R^n \to R$ is a Liapunov function for the vector field $\mathbf{F}$ in the *bounded* open set U; let S_c be a sublevel set of V. Assume the intersection $S_c \cap U$ is closed in R^n. Then $S_c \cap U$ is forward-invariant under $\mathbf{F}$.

Proof. Let $x \in S_c \cap U$. Since V is non-increasing along solutions, $V(x) \leq c$ implies $V(\phi(x, t)) \leq c$, so $\phi(t, x) \in S_c$ for $t \in [0, \omega_x)$. The problem is that the forward orbit of x might leave the open set U. Arguing by contradiction, suppose this happens, and let $t_0 > 0$ be the smallest $t < \omega_x$ so that $\phi(t, x) \notin U$.

Then there is a sequence $t_k \to t_0$ so that $x_k = \phi(t_k, x)$ is in $S_c \cap U$. Thus x_k converges to the point $\bar{x} = \phi(t_0, x) \in R^n$. As a limit point of a sequence in $S_c \cap U$ (a closed set by hypothesis) we must have $\bar{x} \in S_c \cap U$, in contradiction with the way t_0 was chosen. This shows the forward orbit of x stays in U, for all $t \in [0, \omega_x)$. And since U is bounded, it follows that $\omega_x = +\infty$, and $\phi(t, x) \in S_c \cap U \; \forall t \geq 0$, proving the invariance.

If $\dot{V} = 0$ in all of R^n, the proposition says simply that *any bounded sublevel set of V is forward-invariant*; and *bounded* is needed only to guarantee solutions are defined for all $t \geq 0$. The next easiest case in which the hypothesis is satisfied occurs when the closed set S_c (or its connected component intersecting U) is contained in the bounded open set U.

The original use of Liapunov functions was to characterize stability of equilibria. Recall x_0 is an *isolated equilibrium* for $\mathbf{F}$ if $\mathbf{F}(x_0) = 0$ and there are no other equilibria in some ball $B_r(x_0)$ with center x_0.

Definition. Let x_0 be an isolated equilibrium for the vector field $\mathbf{F}$. A function $V : R^n \to R$ is a Liapunov function for x_0 if (i) V has a strict local minimum at x_0; (ii) V is a Liapunov function for $\mathbf{F}$ in the ball $B_r(x_0)$. V is a *strict* Liapunov function for x_0 if in (ii) we have that V is a strict Liapunov function for $\mathbf{F}$ in $B_r(x_0) \setminus \{x_0\}$.

Liapunov's theorem. If x_0 is an isolated equilibrium for $\mathbf{F}$ and V is a Liapunov function (or strict Liapunov function) for x_0, then x_0 is stable (resp. asymptotically stable).

Proof. For any $r > 0$ small enough, x_0 is the only equilibrium in $B_r(x_0)$, V is a Liapunov function in $B_r(x_0)$ and $V(x) > V(x_0)$ there. Then we can find $c > V(x_0)$ so that the component of the sublevel set for c containing x_0 (which we denote by S_c) is compact and contained in $B_r(x_0)$. We can then find a ball $B_s(x_0)$ so that:

$$B_s(x_0) \subset S_c \subset B_r(x_0).$$

Let $x \in B_s(x_0)$. Then since V is a Liapunov function in $B_r(x_0)$ we have $V(\phi(t, x)) \leq c$ for all $t \geq 0$, so $\phi(t, x) \in S_c$, and in particular is in $B_r(x_0)$, for each $t \geq 0$. This shows x_0 is stable.

If V is a strict Liapunov function in $B_r(x_0)$ and $x \in B_s(x_0)$ is any point different from x_0 $V(\phi(t, x))$ is strictly decreasing in t, hence has a limit $L \geq V(x_0)$. S_c is compact and invariant; let $\bar{x}$ be a point in the ω-limit set $\omega(x)$, so $\phi(t_k, x) \to \bar{x}$ and $V(\phi(t_k, x)) \to L = \inf_{t \geq 0} V(\phi(t, x))$ for some sequence $t_k \to \infty$, so $V(\bar{x}) = L$. If $\bar{x} \neq x_0$ we have $V(\phi(t, \bar{x})) < L$ for $t > 0$ small, while $\phi(t + t_k, x) = \phi(t, \phi(t_k, x))$ is close to $\phi(t, \bar{x})$ for k large, hence is also smaller than L for k large, contradicting the fact that L is the inf. This shows $\omega(x) = \{x_0\}$ for any x sufficiently close to x_0, so x_0 is asymptotically stable.

Example: Gradient systems. Let $V : R^n \to R$ be a smooth function. The vector field

$$\mathbf{F} = -\nabla V$$

is the *gradient system* (or 'gradient flow') associated to V. Then V is a Liapunov function for $\mathbf{F}$ in the whole space ($U = R^n$) and a strict Liapunov function on any open set U that has no critical points of V, since:

$$\nabla V \cdot \mathbf{F} = -||\nabla V||^2 \leq 0.$$

The critical points of V are the equilibria of $\mathbf{F}$. If x_0 is a strict (isolated) local min for V, then V is a stirct Liapunov function on any open ball $B_r(x_0)$ not containing other critical points; and then Liapunov's theorem implies x_0 is asymptotically stable.

It often happens that one can find a Liapunov function, but not a strict one, in a neighborhood of an isolated equilibrium; and still one would like to show that the equilibrium is asympotically stable. This situation is addressed by the next theorem.

We use the notation $\dot{V}(x) = \nabla V(x) \cdot \mathbf{F}(x)$, so $\dot{V} \leq 0$ in U, if V is a Liapunov function for $\mathbf{F}$ in U.

Liapunov-La Salle invariance theorem. Let $V : R^n \to R$ be a Liapunov function for the vector field $\mathbf{F}$ on the bounded open set U. Suppose S_c is a sublevel set of V, and the following hold:

(i) $S_c \cap U$ is closed in R^n;

(ii) There is an isolated equilibrium $x_0 \in S_c \cap U$, and a ball $B_r(x_0)$ contained in $S_c \cap U$.

(iii) Let $Z = \{x \in S_c \cap U; \dot{V}(x) = 0\}$. If $x \in Z$ and its forward orbit $\mathcal{O}^+(x)$ is contained in Z, then $x = x_0$.

Then x_0 is asymptotically stable, and for any $x \in S_c \cap U$, $\phi(t, x) \to x_0$ as $t \to +\infty$.

Proof. From the *proposition* above, we know $S_c \cap U$ is a compact forward-invariant set. We first show that if $x \in S_c \cap U$, then V is constant in $\omega(x)$ (which is a non-empty, compact, forward-invariant subset of S_c). By contradiction, let $x_1 \neq x_2$ be points in $\omega(x)$, with (say) $V(x_1) < V(x_2)$. Then we may find sequences t_k, s_k tending to infinity to that $V(\phi(t_k, x)) \to V(x_1)$ and $V(\phi(s_k, x)) \to V(x_2)$. But then we may find indices k, k' so that $t_{k'} < s_k$ and:

$$V(x_2) - \epsilon < V(\phi(s_k), x) \leq V(\phi(t_{k'}, x) < V(x_1) + \epsilon,$$

where the middle inequality follows from V nonincreasing along solutions. For ϵ small enough, this contradicts $V(x_1) < V(x_2)$. This shows V is constant in $\omega(x)$.

But $\omega(x)$ is non-empty, forward-invariant and contained in the compact set S_c, so if $\bar{x} \in \omega(x)$ the entire positive orbit $\mathcal{O}^+(\bar{x})$ is contained in $\omega(x)$. So V is constant along this orbit, which implies $\dot{V} \equiv 0$ at each point of the forward orbit of $\bar{x}$ (which is contained in $S_c \cap U$). But then the hypothesis implies the forward orbit of $\bar{x}$ consists of the point x_0. This shows $\omega(x) =$

$\{x_0\}$ for each $x \in S_c \cap U$, and since $S_c \cap U$ contains a ball centered at x_0 this means x_0 is asymptotically stable.

Remark: Note the difference between the 'strict Liapunov function' case of Liapunov's theorem and the invariance theorem just proved. The conclusion is the same in both cases; but in the former the hypothesis is $Z = \{x_0\}$, while the hypothesis of the second theorem is weaker. We only require:

$$x \in Z \text{ and } \mathcal{O}^+(x) \subset Z \Rightarrow x = x_0.$$

(Also, in the invariance theorem we don't need to establish that $\{x_0\}$ is an strict local minimum of V.)

Note also that it is part of the conclusion that the *basin of attraction* of $\{x_0\}$ (the set of points $x \in R^n$ whose forward orbit converges to x_0) contains the set $S_c \cap U$.

The statement simplifies if V is a Liapunov function for $\mathbf{F}$ in all of R^n:

Corollary. Suppose $\dot{V} \leq 0$ in R^n and x_0 is an isolated equilibrium. Let S_c be a bounded sublevel set of V containing x_0 (and hence a compact forward-invariant set). With $Z = \{x \in S_c | \dot{V}(x) = 0\}$, suppose the following holds: *the only point of Z whose entire forward orbit is contained in Z is x_0*. Then x_0 is asymptotically stable, and S_c is contained in its basin of attraction.