Math 341 – Spring 2013
Exam 3
solutions

1. (i) $f_n \to f$ $\quad f(x) = \begin{cases} x^2, & 0 \le x < 1 \\ 1/2, & x = 1 \\ 0, & x > 1 \end{cases}$

(ii) No, since f is not continuous at 1.

(iii) $\dfrac{x^2}{1+x^n} < x^{2-n} \le 2^{2-n}$ for $x \ge 2$. So $\sup\limits_{[2,\infty)} |f_n(x)| \le \dfrac{4}{2^n} \to 0$,

so $f_n \to 0$ unif'ly in $[2,\infty)$.

2. (i) $f_n' = \dfrac{1-nx^2}{(1+nx^2)^2} = 0$ for $x = \pm \dfrac{1}{\sqrt{n}}$

so $\sup\limits_{x\in\mathbb{R}} |f_n(x)| = |f_n(\frac{1}{\sqrt{n}})| = \dfrac{1}{2\sqrt{n}} \to 0$, and $f_n \to 0$ unif'ly on $\mathbb{R}$.

(ii) $f_n'(0) = 1$ and $f_n'(x) \sim -\dfrac{nx^2}{n^2x^4} \longrightarrow 0$ if $x \ne 0$. The pointwise limit of f_n' is not cont. at 0.

3. (i) $\dfrac{1}{x^2+n^2} \le \dfrac{1}{n^2}$ $(\forall x \in \mathbb{R})$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ conv., so $\sum\limits_{n=1}^{\infty} \dfrac{1}{x^2+n^2}$ conv.

unif'ly on $\mathbb{R}$, (by the M-test), so h is continuous.

(ii) $\left(\dfrac{1}{x^2+n^2}\right)' = \dfrac{-2x}{(x^2+n^2)^2}$ and $\left[\dfrac{x}{(x^2+n^2)^2}\right]' = \dfrac{(x^2+n^2)^2 - 2x(x^2+n^2)2x}{(x^2+n^2)^4} = 0$

if $x^2+n^2 = 4x^2$, or $x = \pm\dfrac{n}{\sqrt{3}}$. Thus $\dfrac{|x|}{(x^2+n^2)^2} \le \dfrac{n}{\sqrt{3}} \cdot \dfrac{9}{16n^4} = \dfrac{9}{16\sqrt{3}} \cdot \dfrac{1}{n^3}$.

Since $\sum \dfrac{1}{n^3}$ conv, $\sum \left(\dfrac{1}{x^2+n^2}\right)'$ conv. unif'ly on $\mathbb{R}$ (M-test). So h is diff'ble, h' cont. on $\mathbb{R}$.

4. (i) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n}$ diverges for $x \ge 1$, converges for $x = -1$. So $R = 1$.

(ii) $g'(x) = \sum\limits_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = 1 - x + x^2 - x^3 + \dots = \dfrac{1}{1+x}$ $(|x|<1)$ geometric series ratio $= -x$

(iii) If $\sum\limits_{n=0}^{\infty} a_n x^n$ conv. at $x = R$ (or at $x = -R$), then $\sum\limits_{n=0}^{\infty} |a_n||x|^n$ conv. for $|x| < R$.

(and diverges for $|x| > R$).

Bonus $\mathcal{F}$ equicont on $[a,b]$: $(\forall \varepsilon>0)(\exists \delta>0)(\forall f \in \mathcal{F})(\forall x,y \in [a,b])(|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$.

Ascoli–Arzelà thm $\mathcal{F}$ unif'ly bounded and equicontinuous on $[a,b]$

$\Longrightarrow \forall (f_n)_{n\ge1}$ seq. in $\mathcal{F}$ $\exists (f_{n_j})_{j\ge1}$ subsequence, unif'ly convergent on $[a,b]$.