Math 341, Spring 2013
Exam 2/solutions

1. Let $x_0 \in f^{-1}(U)$, so $f(x_0) \in U$. Since U is open, $\exists\, \varepsilon > 0$ s.t. $V_{f(x_0)}(\varepsilon) \subseteq U$. Since f is continuous, $\exists\, \delta > 0$ s.t. $x \in V_\delta(x_0) \Rightarrow f(x) \in V_\epsilon(f(x_0))$. Thus $x \in V_\delta(x_0) \Rightarrow f(x) \in U$, so $V_\delta(x_0) \subset f^{-1}(U)$, showing $f^{-1}(U)$ is open.

2. If $f(0) = 0$ or $f(1) = 1$, we are done. Otherwise, we have $f(0) > 0$ and $f(1) < 1$, so $g(x) = f(x) - x$ satisfies $g(0) > 0$, $g(1) < 0$. Since g is continuous in $[0,1]$, by the Intermediate Value Theorem $\exists\, c \in (0,1)$ s.t. $g(c) = 0$, or $f(c) = c$.

3. (i) $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = \lim\limits_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0$, since $\left|\cos\left(\frac{1}{x}\right)\right| \leq 1 \;\forall x \neq 0$.

Thus f is diff-ble at 0, with $f'(0) = 0$.

(ii) For $x \neq 0$, from Calculus: $f'(x) = 2x\cos\left(\frac{1}{x}\right) + x^2\left(\frac{1}{x^2}\right)\sin\left(\frac{1}{x}\right)$
$= \sin\left(\frac{1}{x}\right) + 2x\cos\left(\frac{1}{x}\right)$

$\lim\limits_{x \to 0} 2x\cos\left(\frac{1}{x}\right) = 0$ (see above) but $\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist. So f' is not cont. at 0

4. Let $x, y \in [a,b]$ with $x < y$. By the Mean Value Theorem, there exists $z \in (x,y)$ s.t. $f(x) - f(y) = f'(z)(x-y)$. Since $x - y < 0$ and $f'(z) > 0$, we have $f(x) < f(y)$. So f is strictly increasing.

5. (i) f Lipschitz: $\exists\, M > 0$ s.t. $(\forall x, y \in A)\ (|f(x) - f(y)| \leq M|x-y|)$.
Thus if $\epsilon > 0$ is arbitrary, letting $\delta = \frac{\epsilon}{M}$: if $|x-y| < \delta$, $x, y \in A$, then $|f(x) - f(y)| < \epsilon$. So f is unif'ly continuous on A.

(ii) f is unif'ly cont in $[0,1]$, since f is cont. and $[0,1]$ is <u>compact</u>.
$\left|\dfrac{f(x) - f(0)}{x - 0}\right| = \dfrac{1}{\sqrt{x}} \leq M$ leads to contradiction if $x < \frac{1}{M^2}$, so f is not Lipschitz.