MATH 341, ANALYSIS I-THIRD TEST, April 22, 2013.

Instructions. Closed book, closed notes, no electronics. Time given: 50 min. **10 pts per item**. *Full details for full credit.*

1. Consider $f_n : [0, \infty) \to \mathbb{R}$, $f_n(x) = \frac{x^2}{1+x^n}$.

(i) Compute the pointwise limit $f : [0, \infty) \to \mathbb{R}$.

(ii) Does $f_n \to f$ uniformly on $[0, \infty)$? Justify.

(iii) Does $f_n \to f$ uniformly on $[2, \infty)$? Justify.

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by: $f_n(x) = \frac{x}{1+nx^2}$.

(i) Prove that $f_n \to 0$ uniformly on $\mathbb{R}$. (Hint: use Calculus to show that $\sup_{\mathbb{R}} |f_n| = \frac{1}{2\sqrt{n}}$).

(ii) Is the pointwise limit of the derivatives $f'_n(x)$ continuous on $\mathbb{R}$? For which x do we have $\lim f'_n(x) = 0$?

3. Let $h(x) = \sum_{n \geq 1} \frac{1}{x^2+n^2}$.

(i) Show that $h(x)$ is continuous on $\mathbb{R}$.

(ii) Is h differentiable on $\mathbb{R}$? If so, is h' continuous? Justify.

4. Recall the set of x where a power series $\sum a_n x^n$ converges is either all of $\mathbb{R}$, or an interval consisting of $(-R, R)$ (for some $R > 0$) and possibly one or both of its endpoints.

(i) What is R for the power series $g(x) = \sum_{n \geq 1} (-1)^{n-1} \frac{x^n}{n}$? Justify.

(ii) Find an expression for $g'(x)$ (not as a power series), including its domain.

(iii) If R is finite (for a general power series $\sum a_n x^n$), explain why there are at most two points where the series converges conditionally.

Bonus question. State the Ascoli-Arzelà theorem, for a family $\mathcal{F}$ of functions defined on an interval $[a, b]$. Include the definition of "$\mathcal{F}$ is equicontinuous on $[a, b]$".

Remark. The bonus question will replace one item in another question, if answered correctly (no partial credit). Points not transferable to another test!