MATH 341, ANALYSIS I-SECOND TEST, Mar. 20, 2013.

Instructions. Closed book, closed notes, no electronics. Time given: 50 min. 20 pts per problem. *Full details for full credit.*

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that if $U \subset \mathbb{R}$ is an open set, the preimage $f^{-1}(U)$ is also an open set.

2. Let $f : [0,1] \to [0,1]$ be continuous. Prove that f must have a fixed point in $[0,1]$ (that is, a point $c \in [0,1]$ so that $f(c) = c$).

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by:

$$f(x) = x^2 \cos(\frac{1}{x}), x \neq 0; \quad f(0) = 0.$$

(i) Show that f is differentiable at $x = 0$;
(ii) Is f' continuous at $x = 0$? Justify.

4. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), with $f' > 0$ on (a,b). Show that f is strictly increasing on (a,b).

5. (i) Show that if $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz on a set $A \subset \mathbb{R}$, then f is uniformly continuous on A. (Include the definitions of "Lipschitz" and "uniformly continuous".)

(ii) Let $f(x) = \sqrt{x}$, defined in $I = \{x \in \mathbb{R}; x \geq 0\}$. Explain why f is uniformly continuous in $[0,1]$, and show that f is *not* Lipschitz in $[0,1]$.