MATH 300-FINAL EXAM- 12/11/2012

Instructions. Closed books, closed notes. No credit for answers without justification. Time given: 120 min **. EIGHT QUESTIONS.**

1. [13] Write down (symbolically) the negation of the statement:

$$(\forall x) x \neq 2 \to (\exists y)(x + 2y = xy).$$

Which one is true: the statement or its negation? *Justify.* (The universe is $\mathbb{R}$).

2.[13] Let S, T and U be non-empty sets. Prove that:

$$S \setminus (T \cap U) \subset (S \setminus T) \cup (S \setminus U).$$

(Justify each step in the proof.)

3. [13] Prove that the relation R on the set $\mathbb{Z} \setminus \{0\}$ of nonzero integers defined by:

$$nRm \leftrightarrow mn > 0$$

is an equivalence relation. Describe all its equivalence classes.

4. [13] Consider the inequality:

$$|\frac{1}{1+x} - (1-x)| < Cx^2.$$

Find a value of $C > 0$ for which this is true for all $x \in \mathbb{R}$ such that $|x| < 1/5$ (and prove that the value of C you choose does work for x in this interval.

5. [12]Let $f : \mathbb{R}_+ \setminus \{1\} \to \{y \in \mathbb{R}; |y| \geq 1, y \neq 1\}$ be defined by:

$$f(x) = \frac{x^2+1}{x^2-1}.$$

(i) Prove that f is a bijection.
(ii) Find the inverse function f^{-1} (including its domain).

6. [12] Let $f, g : \mathbb{R} \to \mathbb{R}, A \subset \mathbb{R}, A \neq \emptyset$. Assume $f(A)$ and $g(A)$ are both bounded above.
(i) Show that $(f+g)(A)$ is bounded above.
(ii) Give an example to show that $\sup_A(f+g)$ may be strictly smaller than $\sup_A f + \sup_A g$.

7. [12] (i) Write down (symbolically) the definition of ' $(x_n)_{n\geq 1}$ is a Cauchy sequence'.

(ii) Let $(x_n)_{n\geq 1}$ be a Cauchy sequence of real numbers. Assume there exists an $M > 0$ so that $x_n \geq M$ for all n. Show that the sequence $(y_n)_{n\geq 1}$ defined by $y_n = \frac{1}{x_n}$ is also a Cauchy sequence.

8. [12] (i) Let $(F, +, \cdot, 0, 1)$ be a field. Define what it means to say "F is an ordered field".

(ii) State the 'Archimedean property' of $\mathbb{R}$.

(iii) State the 'completeness property' of $\mathbb{R}$.

SOURCES OF PROBLEMS:

1. practice test 1
2. exam 1
3. practice test 1
4. exam 2 and oral exam
5. spring 2012 math 300 final (see also fall 2012 exam 1, no.5)
6. practice test 2
7. practice test 2 and oral exam
8. practice test 2 and exam 2