Math 251, spring 2010-Exam 2. Date: 3/25/2010. Justify your answers for credit. Calculators allowed. Closed book, closed notes. Time given: 75 min.

1. Given the ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$\mathcal{B}^{(2)} = \{(2,1),(1,1)\}, \quad \mathcal{B}^{(3)} = \{(1,1,0),(1,-1,0),(0,0,1)\},$$

and the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given in standard coordinates by:

$$T(x_1, x_2, x_3) = (4x_1 + x_3, x_1 - 3x_2),$$

find the matrix of T in the given bases.

2. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the subspace $V = \langle(2,1)\rangle$, parallel to the subspace $K = \langle(1,1)\rangle$ (that is, $Pv \in V, v - Pv \in K$.)

(i) Find the matrix of P with respect to the standard basis of $\mathbb{R}^2$.

(ii) What are the eigenvalues of P and the corresponding eigenspaces? (Justify briefly.)

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator with eigenvalues $x = -2$ and $x = 3$, and corresponding eigenspaces:

$$E(-2) = \langle(1,2,1)\rangle, \quad E(3) = \{x \in \mathbb{R}^3 | x_1 + 2x_2 + x_3 = 0\}.$$

Find matrices P and A in GL_3 so that the matrix of T in the standard basis of $\mathbb{R}^3$ has the form: $[T]_{std} = PAP^{-1}$.

4. Use Gram-Schmidt to find an orthonormal basis for $E^\perp \subset \mathbb{R}^3$, where $E = \langle \frac{1}{3}(2,1,2)\rangle \subset \mathbb{R}^3$.

5. Find matrices A and P in $\mathbb{O}_3$ so that the rotation R by $2\pi/3$ radians (counterclockwise) with axis the one-dimensional subspace E of problem 4 is given in the standard basis by $R = PAP^t$.

Hint. You may use the result of problem 4.

6. (i) Find the least-squares approximate solution to the inconsistent system $Ax = b$:

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

(ii) Using the answer from part (i), find the orthogonal projection of the vector $(1,2,1)$ onto the range of A.