HW 4 sols

**1.**

(i) $\mathrm{Ker}(A^t) \subset \mathbb{R}^4$ $\begin{cases} x_1 + x_3 - x_4 = 0 \\ x_2 + x_3 + 2x_4 = 0 \end{cases}$

$\dim \mathrm{Ker}(A^t) = 4-2 = 2 \rightarrow$ let $x_3 = s$, $x_4 = t$

$x_1 = -s+t$, $x_2 = -s-2t$

basis $\{(-1,-1,1,0), (1,-2,0,1)\}$

Hence

$$\mathrm{Ran}(A) = \{ b \in \mathbb{R}^4 \mid b_3 = b_1 + b_2 \text{ and } 2b_2 = b_1 + b_4 \}$$
$$(\text{or } b_4 = 2b_2 - b_1)$$

(ii) using (i), we replace the zero rows of $A$ rref by two non-zero rows:

$$B = \begin{bmatrix} 1 & 0 & 7/5 \\ 0 & 1 & 1/5 \\ 1 & 1 & 8/5 \\ -1 & 2 & -1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow R_3 = R_1 + R_2 \\ \leftarrow R_4 = 2R_2 - R_1 \end{matrix}$$

**2.**

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/6 & 1/3 & 1/2 \\ 0 & 1 & 0 & -1/3 & 1/3 & -1 \\ 0 & 0 & 1 & 5/6 & -1/3 & 1/2 \end{array}\right], \text{ so } A^{-1} = \frac{1}{6}\begin{bmatrix} -5 & 2 & 3 \\ -2 & 2 & -6 \\ 5 & -2 & 3 \end{bmatrix}$$

**3.**

$$C = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -5 \\ 0 & 3 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 3 \\ 1 & -5 & 1 \end{bmatrix}$$

(the coefficients correspond to COLUMN entries)

**4.**

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 3 & -2 & -1 & -1 \end{bmatrix} \quad AB = \begin{bmatrix} -4 & 10 & 4 & 3 \\ 4 & -10 & -4 & -3 \end{bmatrix}$$

$\mathrm{rk}(AB) = 1$

**5.**

(i) $j^{\text{th}}$ col. of $AC$ is a lin. comb. of the columns of $A$, hence is in $\mathrm{Col}(A)$

Since $\mathrm{Col}(AC)$ is the space spanned by the columns of $AC$, it is contained in $\mathrm{Col}(A)$

(ii) If $v \in \mathrm{Ran}(A)$, $v = Ax$ for some $x \in \mathbb{R}^n$.

Thus $v = (AC)(C^{-1}x) = (AC)\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$ $(\vec{x} = C^{-1}x)$.

So $v \in \mathrm{Ran}(AC)$ (by def'n of range), showing $\mathrm{Ran}(A) \subset \mathrm{Ran}(AC)$

Note $\mathrm{Ran}(AC) = \mathrm{Col}(AC)$ and $\mathrm{Ran}(A) = \mathrm{Col}(A)$, so

it follows that $\mathrm{Ran}(AC) = \mathrm{Ran}(A)$.