Math 251 spring 2010- Final Exam B, 5/7/10 No credit for answers without justification. Closed books, closed notes; calculators allowed. Time given: 120 minutes. **Seven problems**.

1. For the $3 \times 4$ matrix $A$ given below, find:
   (i) [10] Conditions on the coordinates $(b_1, b_2, b_3)$ of $b$ so that the system $Ax = b$ has at least one solution $x \in \mathbb{R}^4$;
   (ii) [10] The general solution of $Ax = b$ if $b = (1, 2, 4)$.
   
   $A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 1 & 2 & 5 \\ -1 & 5 & 6 & 9 \end{bmatrix}$.

2. [5] Let $V = \langle (1, -1, 2, 1), (1, 0, 2, 1) \rangle \subset \mathbb{R}^4$. Find a system of linearly independent defining equations for $V$.

3. [15] Find matrices $A$ and $P$ in $O_3$ so that the rotation $R$ by $\pi/3$ radians (counterclockwise) with axis the one-dimensional subspace $E = \langle (1, 2, 2) \rangle \subset \mathbb{R}^3$ is given in the standard basis by $R = PAP^t$.

4. (i) [10] Find the least-squares approximate solution to the inconsistent system $Ax = b$:
   $\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.
   (ii) [10] Using the answer from part (i), find the orthogonal projection of the vector $(1, 2, 2)$ onto the range of $A$.

5. [15] The characteristic polynomial of the matrix $A$ below is $p(\lambda) = (\lambda - 4)^3$, and $\dim E(4) = 1$. Find $\Lambda$ in ‘standard form’ and $B \in GL_3$ so that $A = BAB^{-1}$.
   
   $A = \begin{bmatrix} 0 & 16 & 0 \\ -1 & 8 & 0 \\ 1 & -3 & 4 \end{bmatrix}$.

6. Given the quadratic form in two variables:
   $Q(x, x) = 2x_1^2 + 4x_1x_2 - x_2^2$, $x \in \mathbb{R}^2$,
   (i) [8] Identify geometrically and sketch the set $\{Q(x, x) = 1\}$;
   (ii) [7] Write down the corresponding diagonal form $Q'(x', x')$. 

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7.[10] Solve the second-order recursion relation (that is, find expressions for $x_n$ as a function of $n$):

$$x_{n+2} = -2x_n + 3x_{n+1}, \quad x_0 = 0, x_1 = 1.$$ 

(Change it to a first-order vector recursion $X_{n+1} = AX_n$ first; then diagonalize $A$)