

**Math 251 fall 2010- Exam 3, 10/23/10.** No credit for answers without justification. Closed books, closed notes. Calculators are allowed. Time given: 75 minutes.

1.(i) Find the least-squares solution of the linear system  $Ax = b$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}; \quad b = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}.$$

(i) Compute the error vector  $e = b - Ax$ , and show it is orthogonal to the range of  $A$ .

2. (i) Find the least-squares straight line fit  $\hat{y} = a + bx$  to the three data points:

$$(x_i, y_i) = (0, 0), (1, 2), (2, 7).$$

(ii) Explain why the residuals  $r_i = \hat{y}(x_i) - y_i$  satisfy  $\sum_{i=1}^3 r_i = 0$ .

3. The characteristic polynomial of the SYMMETRIC matrix:

$$\begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$$

is  $p(\lambda) = \lambda^3 - 27\lambda - 54$ , with roots  $\lambda = 6, \lambda = -3$ .

(i) Find the eigenvalues of  $A$ , and the dimensions of the corresponding eigenspaces.

(ii) Explain how you would find matrices  $D$  (diagonal) and  $P$  orthogonal satisfying  $A = PDP^t$  ( $D$  must be given explicitly; for  $P$ , just explain how you would find its columns.)

4. Find rotated coordinates  $\bar{x}, \bar{y}$  that place the conic below in standard form. Write down the equation in standard form, identify and sketch the conic in the  $(\bar{x}, \bar{y})$  plane and find  $\cos \theta$ , where  $\theta$  is the angle of rotation from the  $(x, y)$  to the  $(\bar{x}, \bar{y})$  axes.

$$2x^2 - 4xy - y^2 + 8 = 0.$$

SOLVE ONLY ONE OF (5.) or (6.) BELOW: (if you choose to address both, points obtained in the problem where you're least successful will be transferred to your Exam 2 grade, but won't count towards Exam 3.)

**5.** Find the general solution of the system of differential equations for  $y_1(t), y_2(t)$  (the answer depends on two arbitrary constants  $c_1, c_2$ ):

$$\begin{aligned}y_1' &= y_1 + 4y_2 \\y_2' &= 2y_1 + 3y_2\end{aligned}$$

**6.** At some initial point in time 100,000 people live in a city and 25,000 people live in its suburbs. Each year 5% of the city population moves to the suburbs and 3% of the suburban population moves to the city.

(i) Find a stochastic matrix and an initial probability vector describing this situation.

(ii) Over the long term, how will the population be distributed between the city and the suburbs?