Math 251, exam 1-solutions.

1. Row-reducing the 'augmented matrix', we find:

$$\begin{bmatrix} 1 & 1 & 2 & 5 & 2 & b_3 \\ 0 & 3 & 4 & 7 & 2 & 2b_3 - b_1 \\ 0 & 0 & 0 & 0 & 0 & 2b_1 - 3b_3 + b_2 \end{bmatrix}.$$

Thus the condition for solvability is $3b_3 = 2b_1 + b_2$. Since the last column is $Ae_5 = (2,2,2) = 2(1,1,1)$, we see that $x_p = (1/2)e_5$ solves the system with $b = (1,1,1)$. To find the general solution, we write down the system defining $Ker(A)$, and solve it with x_3, x_4, x_5 as 'free parameters':

$$\begin{cases} x_1 + x_2 + 2x_3 + 5x_4 + 2x_5 &= 0 \\ \quad 3x_2 + 4x_3 + 7x_4 + 2x_5 &= 0 \end{cases} \rightarrow \begin{array}{rl} x_1 = & -(1/3)(2x_3 + 8x_4 + 4x_5) \\ x_2 = & -(1/3)(4x_3 + 7x_4 + 2x_5) \end{array}$$

This gives the general solution (depending on three real parameters t_1, t_2, t_3):

$$x = (0,0,0,0,1/2) + t_1(2,4,-3,0,0) + t_2(8,7,0,-3,0) + t_3(4,2,0,0,-3).$$

2. Since the equations are l.i, we know $dim(V) = 5 - 2 = 3$. To find a basis for V, we just need to find three l.i. vectors in $\mathbb{R}^5$ satisfying both equations. This can be done 'by inspection', by setting two of x_3, x_4, x_5 successively equal to 0 (the other equal to 1) and solving the system for x_1, x_2. We find:

$$\text{basis for } V : \{(-1/2,0,1,0,0),(0,1,0,1,0),(1/2,1,0,0,1)\}.$$

A basis for the two-dimensional space $V^\perp$ can simply be read off from the coefficients of the equations defining V:

$$\text{basis for } V^\perp : \{(2,0,1,0,-1),(0,1,0,-1,-1)\}.$$

3. Form the matrix A with the given vector as *column* vectors. The vector v is in U if the system $Ax = v$ is solvable (and then the coordinates of x give the coefficients in the linear combination). Row-reducing the augmented matrix $[A|v]$, we find the system:

$$\begin{bmatrix} -2 & 0 & 1 & -4 \\ 1 & 1 & 2 & 2 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 + x_2 + 2x_3 &= 1 \\ \quad x_2 + x_3 &= 3 \\ \quad 3x_3 &= -6 \end{cases},$$

which has the unique solution $x_3 = -2, x_2 = 5, x_1 = 1$. Thus $v \in U$ and:

$$(-4, 2, 1, 3) = (-2, 1, -1, 0) + 5(0, 1, 0, 1) - 2(1, 2, -1, 1).$$

4. A basis for the row space can be given by choosing two linearly independent vectors in $\mathbb{R}^3$ satisfying $x_1+x_2+x_3 = 0$, for example $(1, -1, 0), (0, 1, -1)$. These vectors can be taken as the first two row vectors; and then the third row is obtained from the fact that the entries in each *column* must satisfy $y_3 = 2y_1 + y_2$, so $R_3 = 2R_1 + R_2 = (2, -1, -1)$, giving the matrix:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & -1 & -1 \end{bmatrix}.$$

(Of course, many other solutions are possible.)

5. (i) $Ran(A) \subset \mathbb{R}^5$ and $dim(Ran(A)) \leq 3$, hence $Ax = b$ is inconsistent for most b. Since $Ker(A)$ may be different from $\{0\}$, solutions are not necessarily unique.

(ii) Now we know $rank(A) = 3$, so $Ran(A)$ is a 3-dimensional subspace of $\mathbb{R}^5$ and $Ax = b$ is still inconsistent for most b. Since $dim(Col(A)) = 3$, we know $Ker(A) = \{0\}$, so solutions are unique (when they exist.)

Remarks.
Problem 1: HW2, problems 2,3
Problem 2: HW2, problem 4 (discussed in class, 9/23)
Problem 3: HW1, problem 6; 'computational' handout, problem 3.
Problem 4: HW2, problem 9 (discussed in class, 9/23)
Problem 5: handout 'geometry', section 7 (discussed in class, 9/23)