Math 231, fall 2009- Homework set 3.

1. (i) Write down the coordinate transformation associated with the ordered basis of $\mathbb{R}^3$:

$$\mathcal{B} = \{(1,0,-1),(2,1,-1),(0,0,1)\}.$$

(That is, find expressions for x_1', x_2', x_3' in terms of x_1, x_2, x_3).

(ii)Find a defining equation for the subspace

$$V = \{(x_1,x_2,x_3)|x_1 + 2x_2 - 3x_3 = 0\}$$

in the new coordinate system (x_1', x_2', x_3').

2. Find an equation for the curve $x_1x_2 = 1$ in $\mathbb{R}^2$ in the new coordinate system x_1', x_2' defined by the ordered basis $\{(1,1),(1,-1)\}$ of $\mathbb{R}^2$.

3. Find a defining equation for the image of the subspace $V \subset \mathbb{R}^3$ given in problem 1 under P^{-1}, where P is the isomorphism of $\mathbb{R}^3$:

$$x = Px', \quad P(x_1', x_2', x_3') = (x_1' - x_2', x_1' + x_2' + x_3', x_1' + x_3').$$

4. (i) Show that $A^t = -A$, $A \in \mathbb{M}_n$, n odd, implies $det(A) = 0$. (That is, skew-symmetric matrices are never invertible in odd dimensions.)

(ii) Show that if $A^2 + \mathbb{I}_n = 0$, then n must be even.

5. Let $P \in \mathbb{M}_n$ satisfy $P^2 = P$. Show that $det(P)$ equals zero or one, and if it equals one then $P = \mathbb{I}_n$. (Hint: in the latter case, P is invertible.)

6. Show that:

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix} = (a_2 - a_1)(a_3 - a_2)(a_3 - a_1).$$

A similar formula holds in all dimensions. (Hint: column subtraction.) This is called the 'Vandermonde determinant', and is useful in numerical analysis (interpolation.)

7. Use a trigonometric formula to show that, for all x, y, z, w:

$$\begin{vmatrix} \sin x & \cos x & \sin(x+w) \\ \sin y & \cos y & \sin(y+w) \\ \sin z & \cos z & \sin(z+w) \end{vmatrix} = 0.$$

8. Given the ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$\{(1,2),(2,1)\}, \quad \{(1,1,0),(1,-1,0),(0,1,1)\}$$

and the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given in standard coordinates by:

$$T(x_1,x_2,x_3) = (3x_1 + x_2 + x_3, x_1 - 2x_2),$$

find the matrix of T with respect to the given bases.

9. Denote by P_V the orthogonal projection operator onto the subspace $V = \{x_1 + x_2 + 2x_3 = 0\}$ of $\mathbb{R}^3$. Find a basis $\mathcal{B}$ of $\mathbb{R}^3$ 'adapted' to V (that is, $\mathcal{B} = \{v_1, v_2, v_3\}$ with $\{v_1, v_2\}$ a basis of V and $v_3 \in V^\perp$). Then find the matrix of P_V in the basis $\mathcal{B}$.

10. Consider the 3×4 matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix}.$$

Note the relation between rows: $R_3 = R_1 + R_2$.

(i) Find 'adapted' bases for $\mathbb{R}^3$ and $\mathbb{R}^4$, that is:

$$\mathcal{B}^{(4)} = \{v_1, v_2, v_3, v_4\}, \quad Row(A) = \langle v_1, v_2 \rangle, Ker(A) = \langle v_3, v_4 \rangle.$$

$$\mathcal{B}^{(3)} = \{w_1, w_2, w_3\}, \quad Ran(A) = \langle w_1, w_2 \rangle, Ker(A^t) = \langle w_3 \rangle$$

and:

$$Av_1 = w_1, \quad Av_2 = w_2.$$

(ii) With the standard bases, A defines a linear transformation $T_A : \mathbb{R}^4 \to \mathbb{R}^3$. Write down the matrix of T_A in the adapted bases you found in part(i). Justify.