Math 231, fall 2009- Homework set 3.

1. (i) Write down the coordinate transformation associated with the ordered basis of $\mathbb{R}^3$:

$$B = \{(1, 0, -1), (2, 1, -1), (0, 0, 1)\}.$$ (That is, find expressions for $x'_1, x'_2, x'_3$ in terms of $x_1, x_2, x_3$).

(ii) Find a defining equation for the subspace $V = \{(x_1, x_2, x_3) | x_1 + 2x_2 - 3x_3 = 0\}$ in the new coordinate system $(x'_1, x'_2, x'_3)$.

2. Find an equation for the curve $x_1x_2 = 1$ in $\mathbb{R}^2$ in the new coordinate system $x'_1, x'_2$ defined by the ordered basis $\{(1, 1), (1, -1)\}$ of $\mathbb{R}^2$.

3. Find a defining equation for the image of the subspace $V \subset \mathbb{R}^3$ given in problem 1 under $P^{-1}$, where $P$ is the isomorphism of $\mathbb{R}^3$:

$$x = Px', \quad P(x'_1, x'_2, x'_3) = (x'_1 - x'_2, x'_1 + x'_2 + x'_3, x'_1 + x'_3).$$

4. (i) Show that $A^t = -A$, $A \in \mathbb{M}_n$, $n$ odd, implies $\det(A) = 0$. (That is, skew-symmetric matrices are never invertible in odd dimensions.)

(ii) Show that if $A^2 + I_n = 0$, then $n$ must be even.

5. Let $P \in \mathbb{M}_n$ satisfy $P^2 = P$. Show that $\det(P)$ equals zero or one, and if it equals one then $P = I_n$. (Hint: in the latter case, $P$ is invertible.)

6. Show that:

$$\begin{vmatrix}
1 & 1 & 1 \\
a_1 & a_2 & a_3 \\
a_1^2 & a_2^2 & a_3^2
\end{vmatrix} = (a_2 - a_1)(a_3 - a_2)(a_3 - a_1).$$

A similar formula holds in all dimensions. (Hint: column subtraction.) This is called the 'Vandermonde determinant', and is useful in numerical analysis (interpolation.)

7. Use a trigonometric formula to show that, for all $x, y, z, w$:

$$\begin{vmatrix}
\sin x & \cos x & \sin(x + w) \\
\sin y & \cos y & \sin(y + w) \\
\sin z & \cos z & \sin(z + w)
\end{vmatrix} = 0.$$
8. Given the ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$:

$\{(1, 2), (2, 1)\}, \quad \{(1, 1, 0), (1, -1, 0), (0, 1, 1)\}$

and the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given in standard coordinates by:

$T(x_1, x_2, x_3) = (3x_1 + x_2 + x_3, x_1 - 2x_2),$

find the matrix of $T$ with respect to the given bases.

9. Denote by $P_V$ the orthogonal projection operator onto the subspace $V = \{x_1 + x_2 + 2x_3 = 0\}$ of $\mathbb{R}^3$. Find a basis $\mathcal{B}$ of $\mathbb{R}^3$ ‘adapted’ to $V$ (that is, $\mathcal{B} = \{v_1, v_2, v_3\}$ with $\{v_1, v_2\}$ a basis of $V$ and $v_3 \in V^\perp$). Then find the matrix of $P_V$ in the basis $\mathcal{B}$.

10. Consider the $3 \times 4$ matrix:

$$A = \begin{bmatrix}
1 & 2 & -1 & 0 \\
0 & 1 & 3 & 1 \\
1 & 3 & 2 & 1 
\end{bmatrix}.$$ 

Note the relation between rows: $R_3 = R_1 + R_2$.

(i) Find ‘adapted’ bases for $\mathbb{R}^3$ and $\mathbb{R}^4$, that is:

$\mathcal{B}^{(4)} = \{v_1, v_2, v_3, v_4\}, \quad \text{Row}(A) = \langle v_1, v_2 \rangle, \quad \text{Ker}(A) = \langle v_3, v_4 \rangle.$

$\mathcal{B}^{(3)} = \{w_1, w_2, w_3\}, \quad \text{Ran}(A) = \langle w_1, w_2 \rangle, \quad \text{Ker}(A^T) = \langle w_3 \rangle$

and:

$Av_1 = w_1, \quad Av_2 = w_2.$

(ii) With the standard bases, $A$ defines a linear transformation $T_A : \mathbb{R}^4 \to \mathbb{R}^3$. Write down the matrix of $T_A$ in the adapted bases you found in part (i). Justify.