**The variation of parameters formula.**

The ‘variation of parameters’ method may be recast as a useful formula to represent a particular solution to a non-homogeneous linear differential equation, as an integral involving the ‘forcing function’ and a well-chosen solution of the corresponding *homogeneous* problem. This is general, but it is described here only in the constant-coefficient case.

For first-order equations we know this already. To solve:

\[ y' + by = f(t), \quad y = y(t) \]

we use \( e^{bt} \) as an integrating factor and obtain:

\[ y_p(t) = \int_0^t e^{-b(t-s)} f(s) ds, \]

the solution satisfying \( y_p(0) = 0 \). Note that \( e^{-bt} \) is the unique solution of the *homogeneous* equation with value 1 at \( t = 0 \). The other thing to observe is that this formula makes sense also for *piecewise continuous* forcing terms \( f(t) \).

Consider now the second-order equation:

\[ ay'' + by' + cy = f(t), \quad y = y(t). \]

Let \( r_1 \neq r_2 \) be the roots of the characteristic equation, which may be complex. (The conclusion stated below also holds for the case of double roots.) Assuming a particular solution of the form:

\[ y_p(t) = e^{r_1 t} u_1(t) + e^{r_2 t} u_2(t) \]

and setting up the usual system for \( u'_1, u'_2 \), we are led to:

\[ u'_1 = \frac{1}{r_1 - r_2} e^{-r_1 t} f(t), \quad u'_2 = -\frac{1}{r_1 - r_2} e^{-r_2 t} f(t), \]

and after integration:

\[ y_p(t) = \int_0^t \frac{e^{r_1 (t-s)} - e^{r_2 (t-s)}}{r_1 - r_2} f(s) ds. \]

Observe that the solution of the *homogeneous* equation given by:

\[ y_s(t) = \frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2} \]
has the initial conditions:

\[ y_s(0) = 0, \quad y_s'(0) = 1. \]

In the case of complex roots \( \alpha \pm i\beta \), it is easy to compute that:

\[ y_s(t) = \frac{1}{\beta} e^{\alpha t} \sin(\beta t) \]

(the subscript \( s \) is supposed to remind one of ‘sine’; in fact it is easy to show that \( y_s(t) \) is always an odd function of \( t \).) If the roots are real, we can always write them in the form \( \alpha \pm \beta \) with \( \alpha, \beta \) real and \( \beta > 0 \), and then:

\[ y_s(t) = \frac{1}{\beta} e^{\alpha t} \sinh(\beta t) \]

(using the hyperbolic sine.)

Thus the representation formula in the second-order case may be written as:

\[ y_p(t) = \int_0^t y_s(t-u)f(u)du. \]

Verify that \( y_p(t) \) is the solution of the non-homogeneous problem with initial conditions \( y_p(0) = y_p'(0) = 0 \).

Remark. Note again that the integral is perfectly well-defined for piecewise continuous forcing term \( f(t) \).

Notation. Functions defined by an integral of two other functions (of the above type) are known as convolution products, denoted by a star:

\[ (f \ast g)(t) = \int_0^t f(t-u)g(u)du. \]

The convolution product has many of the properties of the product of real numbers, except that there is no function ‘1’ (that is, which convolved with any function \( f \) reproduces \( f \).) For example, it is commutative:

\[ f \ast g = g \ast f, \]

as one verifies easily using a change of variables in the integral.

Finally, denoting by \( y_c(t) \) the solution of the homogeneous equation with IC \( y_c(0) = 1, y'_c(0) = 0 \) (as in ‘cosine’), we have the ‘formula’ for the solution of the IVP for the non-homogeneous equation with IC \( y(0) = y_0, y'_0 = y_1 \):

\[ y(t) = y_0 y_c(t) + y_1 y_s(t) + (y_s \ast f)(t). \]