The variation of parameters formula.

The 'variation of parameters' method may be recast as a useful formula to represent a particular solution to a non-homogeneous linear differential equation, as an integral involving the 'forcing function' and a well-chosen solution of the corresponding *homogeneous* problem. This is general, but it is described here only in the constant-coefficient case.

For first-order equations we know this already. To solve:

$$y' + by = f(t), \quad y = y(t)$$

we use e^{bt} as an integrating factor and obtain:

$$y_p(t) = \int_0^t e^{-b(t-s)} f(s) ds,$$

the solution satisfying $y_p(0) = 0$. Note that e^{-bt} is the unique solution of the *homogeneous* equation with value 1 at $t = 0$. The other thing to observe is that this formula makes sense also for *piecewise continuous* forcing terms $f(t)$.

Consider now the second-order equation:

$$ay'' + by' + cy = f(t), \quad y = y(t).$$

Let $r_1 \neq r_2$ be the roots of the characteristic equation, which may be complex. (The conclusion stated below also holds for the case of double roots.) Assuming a particular solution of the form:

$$y_p(t) = e^{r_1 t} u_1(t) + e^{r_2 t} u_2(t)$$

and setting up the usual system for u_1', u_2', we are led to:

$$u_1' = \frac{1}{r_1 - r_2} e^{-r_1 t} f(t), \quad u_2' = -\frac{1}{r_1 - r_2} e^{-r_2 t} f(t),$$

and after integration:

$$y_p(t) = \int_0^t \frac{e^{r_1(t-s)} - e^{r_2(t-s)}}{r_1 - r_2} f(s) ds.$$

Observe that the solution of the *homogeneous* equation given by:

$$y_s(t) = \frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2}$$

has the initial conditions:

$$y_s(0) = 0, \quad y_s'(0) = 1.$$

In the case of complex roots $\alpha \pm i\beta$, it is easy to compute that:

$$y_s(t) = \frac{1}{\beta} e^{\alpha t} \sin(\beta t)$$

(the subscript s is supposed to remind one of 'sine'; in fact it is easy to show that $y_s(t)$ is always an odd function of t.) If the roots are real, we can always write them in the form $\alpha \pm \beta$ with α, β real and $\beta > 0$, and then:

$$y_s(t) = \frac{1}{\beta} e^{\alpha t} \sinh(\beta t)$$

(using the hyperbolic sine.)

Thus the representation formula in the second-order case may be written as:

$$y_p(t) = \int_0^t y_s(t-u) f(u) du.$$

Verify that $y_p(t)$ is the solution of the *non-homogeneous* problem with initial conditions $y_p(0) = y_p'(0) = 0$.

Remark. Note again that the integral is perfectly well-defined for piecewise continuous forcing term $f(t)$.

Notation. Functions defined by an integral of two other functions (of the above type) are known as *convolution products*, denoted by a star:

$$(f * g)(t) = \int_0^t f(t-u) g(u) du.$$

The convolution product has many of the properties of the product of real numbers, except that there is no function '1' (that is, which convolved with any function f reproduces f.) For example, it is commutative:

$$f * g = g * f,$$

as one verifies easily using a change of variables in the integral.

Finally, denoting by $y_c(t)$ the solution of the *homogeneous* equation with IC $y_c(0) = 1, y_c'(0) = 0$ (as in 'cosine'), we have the 'formula' for the solution of the IVP for the non-homogeneous equation with IC $y(0) = y_0, y_0' = y_1$:

$$y(t) = y_0 y_c(t) + y_1 y_s(t) + (y_s * f)(t).$$