Math 231 spring 2009- Exam 2, 3/12/09. No credit for answers without justification. Closed books, closed notes. Calculators are NOT allowed. Time given: 75 minutes. **(Five problems, 20 pts each).**

1. Consider the non-homogeneous equation:

$$y'' + 3y' + 2y = 10 + \cos(2t), \quad y = y(t).$$

Find the steady-state solution, its average value, period and amplitude of the oscillations about the average value.

2. Use the 'variation of parameters' method to find **ONE** solution of the non-homogeneous equation:

$$x^2 y'' + xy' - 4y = x^3, \quad y = y(x), x > 0.$$

(***Remember to change to 'standard form' first!***) Use the fact that

$$y_1(x) = x^2, \quad y_2(x) = x^{-2}$$

are solutions of the corresponding homogeneous equation.

3. A particle of mass 1 kg moves on a line under a ***repelling*** force proportional to distance to the origin; the force equals 16 N when the particle is at $x = 1$ m.

(i) Write down the differential equation of motion for the position $x(t)$ and its general solution (***note: this is not simple harmonic motion***);

(ii) Assume $x(0) = -1$ and $x'(0) = v_0 > 0$. Find the condition on $v_0 > 0$ that guarantees the particle eventually goes past the origin. (*Hint:* Find the solution $x(t)$ with these initial conditions, and then find the condition v_0 must satisfy for the equation $x(t) = 0$ to have a solution t.)

4. Consider the first-order system for $X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} X' + \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(i) Find the general solution. Give your answer in vector form, so that the terms depending on an arbitrary constant (if any) have the form: (*function of* t) times (*constant vector*).

(ii) Show that, as $t \to +\infty$, all solutions are asymptotic to the same line through the origin (in the x, y plane.)

5. (i) Solve the first-order initial-value problem:

$$y' + 3y = f(t) = \begin{cases} 1, & 0 \le t < 2 \\ 0, & 2 \le t < 4 \\ 1, & t \ge 4 \end{cases}, \quad y(0) = 0.$$

Your answer must be explicit in each range (i.e., it may not contain 'step functions'), but it may be given in terms of the function:

$$y_\theta(t) = \frac{1}{3}(1 - e^{-3t}).$$

(*Hint*: you may express $f(t)$ in terms of 'step functions', and then use a known 'formula'; or you may use the variation-of-parameters formula directly.)

(ii) Does $y(t)$ have a finite limit as $t \to \infty$? (If so, find the limit.)