Examples of first-order systems.[1]

A general system of two linear first-order differential equations in two variables $x(t), y(t)$ (with constant coefficients) has the form:

$$AX' + BX = F(t), \quad X = X(t), \quad X(0) = X_0 \in \mathbb{R}^2,$$

where $F(t) = (f_1(t), f_2(t))$ and the unknown $X(t) = (x(t), y(t))$ take values in $\mathbb{R}^2$ and A, B are constant 2×2 matrices. The system is in 'standard form' when A is the identity matrix:

$$X' + BX = F(t), \quad X = X(t), \quad X(0) = X_0.$$

Systems in standard from have the property that the general solution depends on *two* arbitrary constants, and the initial-value problem has a unique solution, for any $X_0 \in \mathbb{R}^2$. Both facts fail for the general system: the general solution may depend on only one parameter (or even zero parameters), and only *some* initial-value problems are solvable.

Clearly if A is invertible we may multiply the equation on the left by A^{-1} to reduce it to 'standard form'. Thus, whenever A is invertible, the system has the same general features as a 'standard form' system.

Example 1. $2x' - x + y' + 4y = 1, \quad x' - y' = t - 1.$
In vector form:

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} X' + \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 1 \\ t-1 \end{bmatrix}.$$

To solve it, replace the first equation by its sum with the second (to eliminate y'), then differentiate the new 1st equation and substitute the value of y' from the 2nd equation to obtain a second-order equation in x only; the same method solves the other examples below.

Here A is invertible, so the general solution depends on two arbitrary constants, and all IVPs are solvable. In vector form, it is:

$$X = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1/4 \end{bmatrix} + \begin{bmatrix} \frac{2}{3}t^2 - \frac{7}{3}t \\ \frac{1}{6}t^2 - \frac{4}{3}t + \frac{7}{4} \end{bmatrix}.$$

We see that, as $t \to \infty$:

$$X(t) \sim t^2 \begin{bmatrix} \frac{2}{3} \\ \frac{1}{6} \end{bmatrix},$$

[1] This discussion is based on examples from [Tenenbaum-Pollard], sections 31D,E,F

so all solutions are asymptotic to the same half-line, regardless of initial condition.

Example 2. $x' - x + y' + y = 0, \quad 2x' + 2x + 2y' - 2y = t.$
In vector form:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} X' + \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} X = \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

Here A is not invertible (the system is 'degenerate'), so we expect the general solution will depend on fewer than two arbitrary constants. Indeed one finds (in vector form):

$$X = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{t^2}{16} + \frac{t}{8} \\ \frac{t^2}{16} - \frac{t}{8} \end{bmatrix}.$$

In particular, for the initial-value problem to have a solution the initial condition X_0 must lie in the subspace $\{c(1,1); c \in \mathbb{R}\}$ (line through the origin.) All solutions are asymptotic (as $t \to \infty$) to the half of the same line lying in the first quadrant.

Example 3. $x' + 4x + y' = 1, \quad x' - 2x + y = t^2.$
In vector form:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} X' + \begin{bmatrix} 4 & 0 \\ -2 & 1 \end{bmatrix} X = \begin{bmatrix} 1 \\ t^2 \end{bmatrix}.$$

Here A is invertible, so we expect two arbitrary constants in the general solution. In vector form, we obtain:

$$X = c_1 e^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -\frac{t}{2} + \frac{5}{8} \\ t^2 - t + \frac{7}{4} \end{bmatrix}.$$

Clearly all IVPs are solvable (choosing c_1 and c_2 appropriately). Moreover we see that, as $t \to \infty$:

$$X(t) \sim c_1 e^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

that is, solutions are asymptotic to the subspace:

$$E^u = \{c(1,-2); c \in \mathbb{R}\}$$

(u for 'unstable.) As $t \to -\infty$, solutions are asymptotic to the subspace:

$$E^s = \{c(1,3); c \in \mathbb{R}\}.$$

(s for 'stable' .)

Example 4. $x' + 3x + y' + y = e^t, \quad x' + x + y' - y = t.$
In vector form:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X' + \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} X = \begin{bmatrix} e^t \\ t \end{bmatrix}.$$

This system is degenerate (A is not invertible), so we expect something 'different' will happen. Indeed, the 'general' solution has *no* arbitrary constants, which means this system has *only one* solution, with initial condition $X(0) = (1/4, 1/4)$ (no other IVPs are solvable.) The solution is:

$$X = \frac{1}{4} \begin{bmatrix} t+1 \\ 2e^t - 3t - 1 \end{bmatrix}.$$