Math 231 spring 2009- Final Exam, 4/30/09 and 5/5/09 (Sections 3 and 5.) No credit for answers without justification. Closed books, closed notes, no calculators. Time given: 120 minutes. **(Eight problems).**

1.[6+6] Consider the linear initial-value problems for $y(x)$.

$(i)\quad y' + 3y = -6, \quad y(0) = 0.$ Find the solution, sketch its graph.

$$(ii) \quad (4 - x^2)y' + y\sqrt{2x+3} = \ln|x|, \quad y(-1/2) = 2.$$

Find the domain of the solution, without solving the equation.

2.[13] Solve the following initial-value problem for a Bernouilli equation (explicit solution). Include the domain of the solution (recall this must be an open interval containing 0, possibly all of $\mathbb{R}$).

$$y' + y = xy^2, \quad y = y(x), \quad y(0) = 1.$$

3.[6+6] Consider the exact equation (no need to check it is exact!)

$$(\frac{2xy-1}{y})dx + (\frac{y+x}{y^2})dy = 0. \quad (y > 0)$$

(i) Find the general solution in implicit form $E(x, y) = c$.

(ii) Find an implicit equation for the integral curve through the point $(x_0, y_0) = (2, 1)$

4. [13]Use the 'variation of parameters' method to find **ONE** solution of the non-homogeneous equation:

$$y'' + \frac{1}{x}y' - 4\frac{y}{x^2} = x, \quad y = y(x), x > 0.$$

Given: $y_1(x) = x^2, \quad y_2(x) = x^{-2}$ are solutions of the corresponding homogeneous equation. Recall the method finds a solution of the form $y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ by setting up a linear system for u_1', u_2'.

5.[13] Consider the first-order system:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} X' + \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Find the general solution. Give your answer in vector form, so that the terms depending on an arbitrary constant (if any) have the form: (*function of t*) times (*constant vector*).

6.[6+6] A particle of unit mass moves on a line, according to the autonomous conservative equation:

$$y'' = 1 - 4y^2, \quad y = y(t).$$

(i) Find the potential $U(y)$ and sketch its graph; find and classify the equilibria (as stable or unstable).

(ii) If $y(t)$ describes a particular motion with initial conditions $y(0) = 2$ and $y'(0) = 0$, find the total energy and the range of the motion. (That is, find the interval occupied by the particle for $t \geq 0$, which may be unbounded.)

7.[13] Find the solution of the non-linear second-order initial-value problem:

$$y'' = 2yy', \quad y = y(t), \quad y(0) = 1, y'(0) = 2.$$

Include the domain of the solution with your answer. (*Hint:* Let $y' = v$; then $y'' = v\frac{dv}{dy}$ leads to a first-order equation for $v(y)$.)

8.[6+6](i) Use Laplace transforms and convolution to give a formula for the solution to the initial-value problem:

$$y'' + 3y' + 2y = g(t), \quad y = y(t), \quad y(0) = 0, y'(0) = 1,$$

where $g(t)$ is an arbitrary piecewise-continuous function for $t \geq 0$. (The answer should be given in the original variable t.)

(ii)Now let $g(t) = \sin(4t)$. Find the steady-state solution $y(t)$ explicitly. (Use any method you want.) Compute its amplitude and period.

BRIEF TABLE OF LAPLACE TRANSFORMS.

$f(t)$	$F(s)$
t	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$e^{at}f(t)$	$F(s-a)$
$\theta(t-a)f(t-a)$	$e^{-as}F(s)$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$