Linear differential equations with discontinuous forcing terms

Many external effects are modeled as forces starting to act on the system 'instantaneously' at some positive time, either persisting or being withdrawn 'suddenly'. It seems reasonable to model these situations by forcing terms with 'jump discontinuities' in the differential equation $L[y] = f(t), y = y(t)$. In these notes we describe some first- and second-order examples of this, based on the variation of parameters formula and the Heaviside unit step function. For simplicity, we treat only constant coefficient operators. *In these notes all functions are taken with domain* $\{t \geq 0\}$.

First order equations. The variation of parameters formula gives for the solution of the initial-value problem:

$$L[y] = y' + by = f(t), \quad y = y(t), \quad t > 0, \quad y(0) = 0$$

the integral expression:

$$y_p(t) = e^{-bt} \int_0^t e^{bs} f(s) ds,$$

where the integral is perfectly well-defined if $f(t), t \geq 0$ has 'jump discontinuities' (finite one-sided limits) at isolated points. A function $y(t), t \geq 0$ is a *solution* if (i) $y(t)$ is differentiable and satisfies the DE in each open interval where f is continuous; (ii) $y(t)$ is continuous (but usually not differentiable) at the points of discontinuity of f. The function defined by the above integral has both properties, and then (as usual) the solution of the DE with $y(0) = y_0$ is given by:

$$y(t) = y_0 e^{-bt} + y_p(t).$$

The simplest problem of this type has right-hand side $f(t)$ given by 'Heaviside's unit step function at $T \geq 0$':

$$\theta(t - T) = \begin{cases} 0, & 0 \leq t < T \\ 1, & t \geq T \end{cases}$$

(It is not really important how the function is defined at the discontinuity, but we take our functions to be right-continuous for definiteness.) The notation is justified by the fact that $\theta(t - T)$ is the translate (by T units, to the right) of the 'unit step function at $t = 0$', denoted by $\theta(t)$.

Example 1. $y' + by = \theta(t-T), \quad y(0) = 0$. The solution is:

$$y(t) = \begin{cases} 0, & 0 \le t < T \\ y_\theta(t-T), & t \ge T \end{cases}$$

Here $y_\theta(t)$ solves the problem:

$$L[y] = y' + by = 1 \text{ for } t > 0, \quad y(0) = 0,$$

and is given explicitly by:

$$y_\theta(t) = \frac{1}{b}(1 - e^{-bt}).$$

This example may be summarized in the form:

$$f(t) = \theta(t-T) \Rightarrow y(t) = \theta(t-T)y_\theta(t-T),$$

which may be thought of as a 'formula', as we'll see later.

Example 2. $y'+by = f(t) = \begin{cases} 0, & 0 \le t < T_1 \\ 1, & T_1 \le t < T_2, \\ 0 & t \ge T_2 \end{cases}$ This represents a 'unit impulse' that starts at $t = T_1$ and ends at $t = T_2$. How does the system react?

First solution. Applying the variation of parameters formula directly we find:

$$y(t) = \begin{cases} 0, & 0 \le t \le T_1 \\ \frac{1}{b}(1 - e^{-b(t-T_1)}), & T_1 \le t \le T_2 \\ \frac{1}{b}e^{-bt}(e^{bT_2} - e^{bT_1}), & t \ge T_2. \end{cases}$$

Second solution. Note that $f(t)$ may be written as:

$$f(t) = \theta(t-T_1) - \theta(t-T_2),$$

and then the solution follows immediately from linearity and the 'formula' given in Example 1:

$$y(t) = \theta(t-T_1)y_\theta(t-T_1) - \theta(t-T_2)y_\theta(t-T_2).$$

Exercise. Verify that both expressions define the same function.

Philosophical discussion. Which solution is better? The second solution was immediate, given by a formula: no integrals to compute and very little work (other than that needed to express $f(t)$ in terms of the Heaviside

function). Many people like 'formulas', and indeed this is the quickest way to obtain a correct expression for the solution. On the other hand, to use the first method we wouldn't need to 'memorize' any formulas (other than variation-of-parameters, which is basic), and it yields an expression that is easy to understand (and plot 'by hand'), and from which one can easily answer the question: *what is the behavior of the solution as $t \to \infty$?* (The answer is not as immediate from the expression obtained by the second method.)

Example 3. $y'+2y = f(t) = \begin{cases} 2, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ -1, & t \ge 2. \end{cases}$, $y(0) = 0$. The solution obtained by direct use of the variation-of-parameters formula is:

$$y(t) = \begin{cases} 1 - e^{-2t}, & 0 \le t \le 1 \\ \frac{1}{2}(1 - 2e^{-2t} + e^{-2(t-1)}), & 1 \le t \le 2 \\ \frac{1}{2}[e^{-2t}(2e^4 + e^2 - 2) - 1], & t \ge 2. \end{cases}$$

It is easy to see that $y(t) \to -1/2$ as $t \to \infty$; in particular $y(t_0) = 0$, for a unique $t_0 > 2$.

Exercise. Verify that the function $y(t)$ is continuous at the points $t = 0, 1, 2$.

Exercise. Find an expression for $f(t)$ in terms of step functions (*Ans:* $f(t) = 2 - \theta(t-1) - 2\theta(t-2)$) and use it to solve the same problem. Can you determine the asymptotic behavior from this second expression for the answer?

Now consider the more general problem:

$$y' + by = \begin{cases} 0, & 0 \le t < T \\ h(t-T), & t \ge T \end{cases}, \quad y(0) = 0.$$

Using the variation of parameters formula and a change of variables in the integral, we find:

$$y(t) = \begin{cases} 0, & 0 \le t \le T \\ y_h(t-T), & t \ge T, \end{cases}$$

where $y_h(t)$ is the solution to the problem:

$$y' + by = h(t) \text{ for } t > 0, \quad y(0) = 0.$$

This result can also be thought of as a 'formula', generalizing the one we had previously found:

$$f(t) = \theta(t-T)h(t-T) \Rightarrow y(t) = \theta(t-T)y_h(t-T).$$

Example 4. $y' + 2y = f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ 0, & t \ge 2, \end{cases}$ with initial condition $y(0) = 0$.

Write the forcing term using step functions (this requires some manipulation of expressions):

$$f(t) = \theta(t-1)(t-1) - \theta(t-2)(t-2) - \theta(t-2).$$

Thus the solution is:

$$y(t) = \theta(t-1)y_h(t-1) - \theta(t-2)[y_h(t-2) + y_\theta(t-2)],$$

or explicitly:

$$y(t) = \begin{cases} 0, & 0 \le t \le 1 \\ y_h(t-1), & 1 \le t \le 2 \\ y_h(t-1) - y_h(t-2) - y_\theta(t-2), & t \ge 3. \end{cases}$$

Here $y_\theta(t) = (1/2)(1-e^{-2t})$ and $y_h(t)$ solves the non-homogeneous problem:

$$y' + 2y = t \text{ for } t > 0, \quad y(0) = 0.$$

This is easily solved by 'undetermined coefficients' to give:

$$y_h(t) = \frac{1}{4}(e^{-2t} + 2t - 1).$$

Remark. If we want to answer the question 'what is the behavior of the solution as $t \to \infty$?', this expression is not very useful. It is easier to use the variation of parameters formula directly, which gives for $t \ge 2$:

$$y(t) = e^{-2t} \int_1^2 e^{2s}(s-1)ds,$$

so we see immediately that $y(t) \to 0$ as $t \to \infty$.

PROBLEMS. (In the problems below, all functions are taken with domain $\{t \geq 0\}$.) The answers must not contain 'step functions', and all functions y_θ or y_h involved must be computed explicitly.

Problem 1. Solve the initial-value problem:

$$y' + 3y = f(t) = \begin{cases} 1, & 0 \leq t < 2 \\ 0, & 2 \leq t < 3 \\ 1, & t \geq 3 \end{cases}, \quad y(0) = 3.$$

Sketch the graph of the solution. Does $y(t)$ have a limit as $y \to \infty$?

Problem 2. Solve with $y(0) = 0$:

$$y' + y = \begin{cases} e^{-t}, & 0 \leq t < 1 \\ 0, t \geq 1 \end{cases}.$$

Sketch the graph of the solution.

Problem 3. (i) Solve the initial-value problem:

$$y' + 3y = f(t) = \begin{cases} 0, & 0 \leq t < 2, \\ 5(t-2), & 2 \leq t < 3, \\ 0, & t \geq 3. \end{cases} \qquad y(0) = 1.$$

(All functions that appear in your solution formula should be computed explicitly; note $y(0) \neq 0$.)

(ii) Describe the behavior of the solution as $t \to \infty$ (use the variation of parameters formula.)

Problem 4. (i)Solve for $t > 0$, with $y(0) = 0$:

$$y' + 2y = f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ (t-1)^2, & 1 \leq t < 3, \\ 0, & t \geq 3. \end{cases}$$

Express your answer in terms of the functions $y_{h_1}(t), y_{h_2}(t), y_\theta(t)$, solutions of:

$$y' + 2y = t^2 \quad (\text{resp. } = t, = 1), \quad y(0) = 0.$$

Hint: To express $f(t)$ in terms of step functions, use the identity:

$$(t-1)^2 = (t-3+2)^2 = (t-3)^2 + 4(t-3) + 4.$$

(ii) Find the functions $y_{h_1}, y_{h_2}, y_\theta$ explicitly. *Ans.* $y_{h_1}(t) = \frac{1}{4}e^{-2t} + \frac{1}{2}(t^2 - t - \frac{1}{2})$. y_{h_2} and y_θ are given above.

(iii) Use the variation of parameters formula (for $t \geq 3$) to find the limit $lim_{t\to\infty} y(t)$.

*Problem 5.*Solve the equation:

$$y' + 2y = f(t) = \begin{cases} t, & 0 \leq t < 1 \\ t-2, & 1 \leq t < 2 \\ 0, & t \geq 2. \end{cases}$$

with initial condition $y(0) = 0$. (The functions $y_h(t)$ that occur should be computed explicitly, and your answer may not contain 'step functions'). What is the behavior of $y(t)$ as $t \to \infty$?

Second-order equations. Recall the variation of parameters formula expresses the solution of a general non-homogeneous problem:

$$L[y] = ay'' + by' + cy = f(t), \quad y = y(t), t > 0, \quad y(0) = y'(0) = 0$$

as the integral:

$$y(t) = \int_0^t y_s(t-u) f(u) du,$$

where $y_s(t)$ is the solution of $ly = 0, y(0) = 0, y'(0) = 1$. For example, if the roots of the characteristic equation are $\alpha \pm i\beta, \beta > 0$:

$$y_s(t) = \frac{1}{\beta} e^{\alpha t} \sin(\beta t),$$

while if they are real and distinct we may write them as $\alpha \pm \beta$ with $\beta > 0$, and then:

$$y_s(t) = \frac{1}{\beta} e^{\alpha t} \sinh(\beta t).$$

The integral makes sense for piecewise-continuous f. We say $y(t)$ is a *solution* if y is twice differentiable and satisfies the equation on each open interval where f is continuous, while y and y' are both continuous at the points of discontinuity of f.

For example, suppose $f(t) = \theta(t-T)$. Then $y(t) = 0$ for $t \in [0, T]$, while for $t \geq T$:

$$y(t) = \int_T^t y_s(t-u) du = \int_0^{t-T} y_s(t-T-v) dv = y_\theta(t-T),$$

where $y_\theta(t)$ solves $L[y] = 1$ for $t > 0$, $y(0) = 0$. Thus we have a formula similar to the first-order case:

$$f(t) = \theta(t - T) \Rightarrow y(t) = \theta(t - T) y_\theta(t - T).$$

Example 5. Solve with $y(0) = y'(0) = 0$:

$$L[y] = y'' + 3y' + 2y = \begin{cases} 0, & 0 \le t < 5 \\ 1, & t \ge 5 \end{cases}$$

Since the roots of the characteristic equation are -2 and -1, or $-\frac{3}{2} \pm \frac{1}{2}$, we have:

$$y_s(t) = 2e^{-3t/2} \sinh(t/2).$$

(This is not needed to solve the problem.) $y_\theta(t)$, the solution of $L[y] = 1$ with zero initial data, is easily found by undetermined coefficients:

$$y_\theta(t) = \frac{1}{2} e^{-2t} - e^{-t} + \frac{1}{2},$$

and then the solution is:

$$y(t) = \begin{cases} 0, & 0 \le t \le 5 \\ y_\theta(t - 5), & t \ge 5. \end{cases}$$

Example 6. Solve with initial conditions $y(0) = y'(0) = 0$:

$$y'' + 3y' + 2y = f(t) = \begin{cases} 1, & 0 \le t < 2 \\ 2, & 2 \le t < 3 \\ 0, & t \ge 3. \end{cases}$$

It is easy to express $f(t)$ in terms of step functions:

$$f(t) = 1 + \theta(t - 2) - 2\theta(t - 3).$$

It follows the solution is given by:

$$y(t) = \begin{cases} y_\theta(t), & 0 \le t \le 2 \\ y_\theta(t) + y_\theta(t - 2), & 2 \le t \le 3 \\ y_\theta(t) + y_\theta(t - 2) + y_\theta(t - 3), & t \ge 3 \end{cases}$$

(for the same function $y_\theta(t)$ as in Example 5.)

With exactly the same argument as in the first-order case (change of variable in the variation of parameters formula) we have for second-order equations:

$$f(t) = \theta(t-T)h(t-T) \Rightarrow y(t) = \theta(t-T)y_h(t-T),$$

where $y_h(t)$ solves $L[y] = h(t)$ for $t > 0$, with zero initial conditions.

Example 7. Solve with initial conditions $y(0) = y'(0) = 0$:

$$L[y] = y'' + 4y' + 5y = f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ 0, & t \ge 2. \end{cases}$$

Writing $f(t)$ in the form:

$$(t-1)\theta(t-1) - (t-2)\theta(t-2) - \theta(t-2),$$

we see the solution is:

$$y(t) = \begin{cases} 0, & 0 \le t \le 1 \\ y_h(t-1), & 1 \le t \le 2 \\ y_h(t-1) - y_h(t-2) - y_\theta(t-2), & t \ge 2. \end{cases}$$

The homogeneous equation has general solution: $e^{-2t}(c_1 \cos t + c_2 \sin t)$; hence the function $y_\theta(t)$, solution of $L[y] = 1$ for $t > 0$ with zero initial conditions, is given by:

$$y_\theta(t) = -e^{-2t}(\frac{1}{5}\cos t + \frac{2}{5}\sin t) + \frac{1}{5}.$$

$y_h(t)$ solves $L[y] = t$ with zero IC, and is easily found by 'undetermined coefficients':

$$y_h(t) = \frac{1}{25}[e^{-t}(4\cos t + 3\sin t) + 5t - 4].$$

PROBLEMS. Solve the following non-homogeneous second-order equations. The answer should not contain the 'step function', and the corresponding functions $y_f(t)$ and $y_\theta(t)$ should be given explicitly.

1.[T-P,p.367]

$$y'' + 2y' + 2y = \begin{cases} e^{-t}, & 0 \le t < 1 \\ 0, & t \ge 1. \end{cases}$$

2.

$$y'' + 3y' + 2y = \begin{cases} 1, & 0 \le t < 2 \\ 2, & 2 \le t < 3 \\ 0, & t \ge 3. \end{cases}$$

3.

$$y'' + 4y' + 5y = \begin{cases} 0, & 0 \le t < 1 \\ (t-1)^2, & 1 \le t < 2 \\ 0, & t \ge 2. \end{cases}$$