
1. For the second-order equation: \((x + 1)y'' - 3xy' + 2y = 0, \ y = y(x)\)
find the general solution, in the form:
\[ y(x) = a_0y_0(x) + a_1y_1(x), \]
where \(y_0(x)\) and \(y_1(x)\) are power series at 0, with coefficients given explicitly up to \(O(x^5)\). What is the interval of convergence?

2. (i) Use Laplace transforms and convolution to give a formula for the solution to the initial-value problem:
\[ y'' - 2y' - 5y = g(t), \ y = y(t), \ y(0) = 0, y'(0) = 1, \]
where \(g(t)\) is an arbitrary piecewise-continuous function for \(t \geq 0\). (The answer should be given in the original variable \(t\).)

(ii) Now let \(g(t) = \begin{cases} 0, & 0 \leq t < 1 \text{ or } t \geq 2; \\ 17, & 1 \leq t < 2 \end{cases}\)

Find the solution \(y(t)\) of the IVP explicitly (your answer should not contain ‘step functions’). Is the solution \(y(t)\) continuous at \(t = 1\) and \(t = 2\)?

3. Find a one-parameter family of solutions for the non-linear equation:
\[ (y + 1)y'' = 3(y')^2, \ y = y(x). \]
(Hint: Let \(y' = v\); then \(y'' = v \frac{dv}{dy}\) leads to a first-order equation for \(v(y)\).)

4. A particle of unit mass moving subject to an attractive central force has trajectory given in polar coordinates by:
\[ r = e^\theta, \ r = r(t), \theta = \theta(t) \quad \text{(a spiral).} \]

Find an expression for the force \(\vec{F}(r) = f(r)\vec{u}_r\) (that is, find \(f(r)\).) Hint: Use \(r^2\theta' = \ell\), a constant of motion (angular momentum) and remember \(f(r) = r'' - r(\theta')^2\).