Math 231 fall 2008- Final Exam, 12/11/08 No credit for answers without justification. Closed books, closed notes. Calculators OK. Time given: 120 minutes. **(Seven problems).**

1. For the autonomous first-order equation given: (i)[5] sketch the diagram of all solutions (including at least two curves in each region defined by the equilibria; indicate vertical asymptotes where needed);

(ii)[5] identify the initial conditions (at $t = 0$) corresponding to solutions defined for all $t \in \mathbb{R}$, and those with solutions defined only on an interval $(-\infty, T_*)$ or (T_*, ∞).

$$y' = y^3 - 9y, \quad y = y(t).$$

2. (i)[10] Show that the equation given below is *exact* and (ii) [10] find the solution with initial condition $y(1) = 0$, in implicit form.

$$(t \sin y - y^2)y' - \cos y = 0, \quad y = y(t).$$

3. [10] A damped oscillator moving under an external periodic force is described by the differential equation:

$$x'' + 2x' + 16x = F \sin(\omega t), \quad x = x(t).$$

Find the amplitude of the steady-state motions as a function of F and ω;

4.[20] Consider the non-homogeneous system for functions $x(t), y(t)$:

$$\begin{cases} 3x' + 3x + 2y = t \\ 4x - 3y' + 3y = e^t \end{cases}$$

Find the general solution of the system.

5.[10] Find the solution of the initial-value problem:

$$y'' - 4y' + 13y = g(t), \quad y = y(t), \quad y(0) = 0, y'(0) = 1,$$

where $g(t) = \begin{cases} 0, & 0 \le t < 1 \text{ or } t \ge 2; \\ A, & 1 \le t < 2 \end{cases}$ (The answer should be given in the original variable t, and should not contain 'step functions'.) How can we define the solution at $t = 1$ so that it is continuous there? (Justify.)

6. A particle of unit mass moves on a line, subject to a force:

$$f(y) = 1 - 3y^2.$$

(i)[10] Find the potential $U(y)$ and sketch its graph; find and classify the equilibria (as stable or unstable).

(ii)[10] If $y(t)$ describes a particular motion with initial conditions $y(0) = 0.75$ and $y'(0) = 0$, find the range of the motion. (That is, find the interval to which the particle's motion is confined.)

7.[10] For the following second-order homogeneous equation ($y = y(t)$) given below, one solution is given. Use the 'reduction of order' method to find a second, linearly independent solution.

$$2t^2y'' + 3ty' - y = 0, \quad y_1 = \frac{1}{t} \quad (t > 0)$$

LAPLACE TRANSFORMS:

$$\mathcal{L}[\sin at](s) = \frac{a}{s^2 + a^2} \quad \mathcal{L}[\cos at](s) = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}[u(t-a)f(t-a)](s) = e^{-as}F(s).$$