Calculus III exercise

Consider the vector field $\vec{\omega}$ defined in the 'punctured plane' $\mathbb{R}^2_* = \mathbb{R}^2 - \{(0,0)\}$:

$$\vec{\omega} = \frac{1}{x^2+y^2}(-y, x).$$

We know the circulation of $\vec{\omega}$ on the unit circle (traversed once, counterclockwise) is 2π, so $\vec{\omega}$ cannot have a potential defined on $\mathbb{R}^2_*$. We also saw that the function

$$f_{LR}(x,y) = \arctan(y/x),$$

defined for $x \neq 0$, is a potential for f, defined in *either* of the two simply-connected domains:

$$D_L = \{(x,y); x < 0\}, \quad D_R = \{(x,y); x > 0\}.$$

(L stands for 'left', R for 'right'.) Recall that $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is the inverse function to $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$.

Consider the connected domain obtained by deleting the negative x-axis from the plane:

$$U = \{(x,y); y \neq 0 \text{ or } x > 0\}.$$

Since $\vec{\omega}$ satisfies $P_y = Q_x$ and U is simply-connected, we know $\vec{\omega}$ has a potential in U. The goal of the exercise is to find such a potential explicitly.

Problem 1. Fix the point $P_0 = (1,0)$ and compute the line integral of $\vec{\omega}$ from P_0 to an arbitrary point (x,y) in U, along the line segment from P_0 to (x,y).

Answer: Denoting by $f_{TB}(x,y)$ the value of the line integral, we have:

$$f_{TB}(x,y) = \arctan\left(\frac{(x-1)^2+y^2+x-1}{y}\right) - \arctan\left(\frac{x-1}{y}\right).$$

This is defined only for $y \neq 0$ (if $y = 0$ and $x > 0$, the line integral is zero-right?) Let $f_{TB}(x,y)$ be the function defined by this expression, in either of the simply-connected domains:

$$D_T = \{(x,y); y > 0\}, \quad D_B = \{(x,y); y < 0\}$$

('top' and 'bottom', of course).

Note that the functions $f_{LR}(x,y)$ and $f_{TB}(x,y)$ 'look' completely different. Are they really? Both have the property that, wherever defined, their gradient coincides with the vector field $\vec{\omega}$. *Therefore, in any connected domain in which both functions are defined, they differ at most by a constant.* The constants in general depend on the particular domain considered. For example, in the first quadrant (which is the intersection $D_T \cap D_R$) the difference $f_{TB}(x, y) - f_{LR}(x, y)$ is *constant*. There is another way to see this, which shows what the possible constants are.

Problem 2. Use the formula for the tangent of the difference:

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

(valid when $\cos\alpha\cos\beta \neq 0$) to show that:

$$\tan f_{TB}(x, y) = y/x = \tan f_{LR}(x, y).$$

Now, if two arcs have the same tangent, their difference is an integer multiple of π. We conclude that, in each connected domain where both are defined:

$$f_{TB}(x, y) - f_{LR}(x, y) = n\pi, \quad n \in \mathbb{Z}.$$

(n may be zero).

The domains in question here are the open quadrants. To find the value of the constant in each quadrant, it is enough to compare the value of the functions at a single point in the quadrant.

Problem 3. Use the points $(1, 1), (1, -1), (-1, -1)$ to show that the constant is zero on the first, third and fourth quadrants.

This shows that, in fact, $f_{TB} = f_{LR}$ in each of these quadrants, so in a sense f_{LR} 'extends' f_{TB} across the positive x axis, where it is not defined. This should surprise you: *the expressions defining $f_{TB}(x, y)$ and $f_{LR}(x, y)$ are completely different, yet in the first, third and fourth quadrants they define the same function!* (Check by plugging in values with a calculator, if you don't believe it). This is a good reminder that a 'function' and 'a particular analytic expression defining a function' are not exactly the same thing.

In fact, we have found a potential $f(x, y)$ for $\vec{\omega}$ defined in U, just by noting that U is the union:

$$U = D_T \cup D_R \cup D_B,$$

namely:

$$f(x, y) = \begin{cases} f_{TB}(x, y), & (x, y) \in D_T \cup D_B \\ f_{LR}(x, y), & (x, y) \in D_R \end{cases}$$

The point is that the reason this definition makes sense is that, in the intersection $(D_T \cup D_B) \cap D_R$, which is the union of the (open) first and fourth quadrants, it doesn't matter which expression you use to compute the value of $f(x, y)$- they give the same result.

The reader who has followed the construction this far might wonder: why can't we continue this game and use f_{LR} in D_L to extend $f(x, y)$ to the union of U and D_L, which is the whole punctured plane? (We know this shouldn't be possible.) If we try that, we get a conflict in the second quadrant- the difference between f_{TB} and f_{LR} is a non-zero constant there:

Problem 4. (i) By comparing their values at $(-1, 1)$, show that on the second quadrant $f_{TB}(x, y) - f_{LR}(x, y) \equiv \pi$.

In particular, you will need to check (using the formula for the tangent of a sum analogous to that given above for the difference) that $\arctan 3 + \arctan 2 = 3\pi/4$.

(ii) Show that $f(-1, -1) - f(-1, 1) = -3\pi/2$, and that $f_{LR}(-1, 1) - f_{LR}(-1, -1) = -\pi/2$. Explain why this confirms that the circulation of $\vec{\omega}$ around the circle with radius $\sqrt{2}$ (traversed once in the clockwise direction) equals -2π.

(*Hint:* draw a diagram- divide the circle into two parts, one contained in U, the other in D_L; then use the potentials f and f_{LR} for $\vec{\omega}$ (defined in U and D_L, respectively).