LINEAR ALGEBRA: SUMMARY OF RESULTS

1. BASIC FACTS

Theorem 1. Let V be finite dimensional. Any finite spanning set can be reduced to a basis.

This follows from the *lemma:* Given a finite linearly dependent set S, one can remove a vector from S without changing its linear span.

Theorem 2. Let V be finite dimensional. Given a linearly independent set I and a finite spanning set S, one always has $card(I) \leq card(S)$.

Corollary 3. If a vector space W contains an infinite linearly independent set, then W is infinite dimensional.

Corollary 4. In a finite dimensional vector space V, any linearly independent set is finite, and can be enlarged to a basis of V. This implies any subspace $U \subset V$ has a *complement*: a subspace $W \subset V$ such that $U \oplus W = V$.

Theorem 5. If V is finite-dimensional and $T \in \mathcal{L}(V)$, then:

$$dim Ker(T) + dim Ran(V) = dim V.$$

Corollary 6. If $dim(V) < dim(W)$, we have $Ran(T) \neq W$ for all $T \in \mathcal{L}(V, W)$ and $Ker(T) \neq \{0\}$ for all $T \in \mathcal{L}(W, V)$.

Corollary 7. If V is finite-dimensional and $T \in \mathcal{L}(V)$, then

$$Ker(T) = \{0\} \Leftrightarrow Ran(T) = V \Leftrightarrow T \text{ is invertible.}$$

Theorem 8. (*change of basis formula*) Let $\mathcal{B} = \{e_1, \ldots e_n\}, \mathcal{B}' = \{f_1, \ldots f_n\}$ be two bases of V, and let $T \in \mathcal{L}(V)$. Denote by $P \in \mathbb{M}_{n\times n}(\mathbb{F})$ the matrix whose column vectors are given by $[f_i]_{\mathcal{B}}, i = 1, \ldots, n$. Then:

$$[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P.$$

Theorem 9. Let V be a finite dimensional vector space over the complex numbers $\mathbb{C}$, $T \in \mathcal{L}(V)$. Then T has an eigenvalue.

Proposition 10. Let $I = \{v_1, \ldots, v_n\}$ be a set of non-zero eigenvectors for $T \in \mathcal{L}(V)$, each with a different eigenvalue. Then I is linearly independent.

Corollary 11. Let V be as in theorem 9, $T \in \mathcal{L}(V)$. Then there exists a basis $\mathcal{B}$ of V so that the matrix $[T]_{\mathcal{B}}$ is upper-triangular, and the set of its diagonal entries is the set of eigenvalues of T.

Corollary 12. Let V be finite-dimensional, $T \in \mathcal{L}(V)$. Then T is diagonalizable if, and only if, the sum of the dimensions of all its eigenspaces equals the dimension of V.

(This follows from Theorem 5 and Proposition 10.)

2. STRUCTURE THEORY (V FINITE-DIMENSIONAL)

Proposition 13. Let $T \in \mathcal{L}(V)$, $dim(V) = n$. Defining the 'generalized kernel' of T by:

$$K_{gen}(T) = \{v \in V | T^j v = 0 \text{ for some } j \geq 1\},$$

we have: $K_{gen}(T) = Ker(T^n)$; in fact $Ker(T^{n_0}) = Ker(T^{n_0+1}) = \dots$ for some $n_0 \leq n$.

Theorem 14. Let $T \in \mathcal{L}(V)$, $\mathcal{B}$ any basis of V in which $[T]_{\mathcal{B}}$ is upper triangular. Then any eigenvalue $\lambda \in \mathbb{F}$ occurs on the diagonal of $[T]_{\mathcal{B}}$ exactly $dim(E_{gen}(\lambda))$ times. In particular, if V is a *complex* vector space, the sum of the algebraic multiplicities of the eigenvalues of T equals the dimension of V.

Definition. Let $T \in \mathcal{L}(V)$. A *chain of length* r for an eigenvalue λ of T is a finite sequence $(v_1, v_2, \dots v_r)$ of vectors in $E_{gen}(\lambda)$ (the generalized eigenspace for λ) such that $Tv_i = \lambda v_i + v_{i+1}$ for $i = 1, \dots r-1$ and $Tv_r = \lambda v_r$. If $\lambda = 0$ is an eigenvalue, a chain for 0 is called a 'null chain'. If the scalar field is $\mathbb{R}$ and $\lambda = a + ib, b \neq 0$, is an eigenvalue of the complexification $T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}})$, a *real chain of length* r for the pair (a, b) is a sequence $(x_1, y_1, \dots, x_r, y_r)$ of vectors x_i, y_i in V so that:

$$Tx_i = ax_i - by_i + x_{i+1}, Ty_i = bx_i + ay_i + y_{i+1}, i = 1, \dots, r-1, \quad Tx_r = ax_r - by_r, Ty_r = by_r + ax_r.$$

Theorem 15. Let $T \in \mathcal{L}(V)$, V a $\mathbb{C}$-vector space. Then V is the direct sum of all the generalized eigenspaces $E_{gen}(\lambda)$ of T. Equivalently, one may write $T = N + D$, where N is nilpotent, D is diagonalizable and N and D commute. (In fact, N and D are unique, and are polynomial functions of T).

Theorem 16.(*Existence of a Jordan basis*) Let $T \in \mathcal{L}(V)$ be a nilpotent operator. Then there exists a basis $\mathcal{B}$ of V which is a disjoint union of r 'null chains' for T. Taking the last vectors in each null chain, we obtain a basis for $Ker(T)$. The matrix of T in $\mathcal{B}$ is said to be in 'Jordan form', and is a direct sum of r 'elementary nilpotent Jordan blocks of sizes d_j', with $d_1 + \dots + d_r = dim(V)$.

Corollary 17. (Jordan form) Let $T \in \mathcal{L}(V)$, where the scalar field is $\mathbb{C}$; let λ be an eigenvalue of T. The generalized eigenspace $E_{gen}(\lambda)$ has a basis $\mathcal{B}$ which is a disjoint union of r chains for λ, of lengths $d_1 \geq \dots \geq d_r$. We have $r = dim(E(\lambda))$ ('geometric multiplicity') and $d_1 + \dots + d_r = dim(E_{gen}(\lambda))$ ('algebraic multiplicity'). The matrix of $T_{|E_{gen}(\lambda)}$ in the basis $\mathcal{B}$ (written in reverse order) is said to be in 'Jordan form'.

Corollary 18. (Real Jordan form) Let $T \in \mathcal{L}(V)$, where the scalar field is $\mathbb{R}$; let $\lambda = a + ib \in \mathbb{C}$ ($b \neq 0$) be an eigenvalue of the complexification $T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}})$. If $(v_1, \dots, v_d)$ is a chain of length d for λ (with $v_j \in V_{\mathbb{C}}$), then letting $x_j = (1/2)(v_j + \bar{v}_j)$ and $y_j = (1/2i)(v_j - \bar{v}_j)$ be the real and imaginary parts of v_j we obtain a real chain of length d $(x_1, y_1, \dots x_d, y_d)$ for (a, b). Doing this for each of the r λ-chains in a Jordan basis for $E_{gen}(\lambda) \subset V_{\mathbb{C}}$ (Corollary 17), of lengths $d_1, \dots d_r$, we obtain a basis $\mathcal{B}$, consisting of r disjoint real (a, b)-chains, for a T-invariant subspace $E(a, b) \subset V$, of dimension $2d_1 + \dots + 2d_r$. The matrix of $T_{|E(a,b)}$ in the basis $\mathcal{B}$ (written in reverse order) is said to be in 'real Jordan form'. The space V is the direct sum of the generalized eigenspaces $E_{gen}(\gamma)$ over all real eigenvalues γ of T, and the spaces $E(a_j, b_j)$, where $a_j \pm ib_j$ ranges over the non-real complex eigenvalues of T.

Proposition 19. Let $T \in \mathcal{L}(V)$ be invertible, where V is a complex vector space. Then, for each r, T has r 'r-th. roots', i.e. operators $S \in \mathcal{L}(V)$ so that $S^r = T$. Each r-th. root is a polynomial in T.

3. Normed spaces and inner-product spaces

Proposition 20. Any norm in $\mathbb{R}^n$ defines a continuous function $\mathbb{R}^n \to [0, \infty)$ (for the usual topology in $\mathbb{R}^n$. As a corollary, any two norms in $\mathbb{R}^n$ are *equivalent.*

Theorem 21. Let $T \in \mathcal{L}(\mathbb{R}^n)$. The sequence:

$$e_N(T) = \sum_{j=0}^{N} \frac{1}{j!} T^j$$

converges to an operator $e^T \in \mathcal{L}(V)$ (in any norm on $\mathcal{L}(\mathbb{R}^n)$). If T, S are commuting operators, we have $e^{(T+S)} = e^T e^S = e^S e^T$. If $S = P^{-1}TP$, then $e^S = P^{-1}e^T P$. The operator-valued function $\Phi(t) = e^{tA}$ is the unique solution of the differential equation $\Phi'(t) = A\Phi(t)$ in $\mathcal{L}(\mathbb{C}^n)$ satisfying the initial condition $\Phi(0) = I$.

Definition 22. A *Banach space* is a complete normed vector space. Some infinite-dimensional examples: (i)the sequence spaces $l^p, 1 \leq p < \infty$, with the usual l^p norms; (ii)the space l^∞ of bounded sequences, with the *sup* norm; (iii) the space $C_b(\mathbb{R})$ of continuous, bounded functions $\mathbb{R} \to \mathbb{R}$, with the *sup* norm. A *Hilbert space* is an inner-product space (with a real or hermitian inner product) which is complete in the corresponding norm. Example: the sequence space l^2 (with the usual inner product).

Proposition 23. A norm (on a real or complex vector space, finite-dimensional) comes from an inner product if and only if it satisfies the parallelogram law.

Proposition 23b. (*Gram-Schmidt*) Given any basis $\{v_1, \ldots v_n\}$ of an inner product space V, one may find an orthonormal basis $\{e_1, \ldots e_n\}$ of V so that $\{e_1, \ldots e_k\}$ and $\{v_1, \ldots v_k\}$ span the same subspace, for each $k = 1, \ldots, n$.

Proposition 24. Recall a *projection* is defined as any operator P such that $P^2 = P$, and an *orthogonal projection* as a projection P such that, for all $v \in V$, $v - Pv \in (RanP)^\perp$. Equivalently, $RanP = (KerP)^\perp$ characterizes orthogonal projections among all projections. If $W \subset V$ is a subspace and $P \in \mathcal{L}(V)$ is orthogonal projection onto W, then $w = Pv$ solves the problem: minimize $||v - w||$ among all $w \in W$.

*Proposition 25.*If $\varphi : V \to \mathbb{F}$ is a linear functional, there exists a unique $w \in V$ such that $\varphi(v) = \langle v, w \rangle$, for all $v \in V$.

This leads to the definition of *adjoint* of an operator T, via $\langle Tv, w \rangle = \langle v, T^* w \rangle$. On any orthonormal basis $\mathcal{B}$ of V, the matrix $[T^*]_\mathcal{B}$ is the conjugate-transpose ('hermitian adjoint') of $[T]_\mathcal{B}$. We have $Ker(T^*) = (RanT)^\perp$ and $Ran(T^*) = (KerT)^\perp$. For two operators T, S, $(ST)^* = T^* S^*$.

Definition/proposition 26. An operator T is *normal* if $T^*T = TT^*$, *self-adjoint* if $T = T^*$. T is normal if and only if $||Tv|| = ||T^*v||$ for all v.

Proposition 27. All eigenvalues of a self-adjoint operator are *real.* If $\lambda \in \mathbb{C}$ is an eigenvalue of a normal operator, then so is $\bar{\lambda}$. For any normal operator, eigenspaces corresponding to distinct eigenvalues are orthogonal.

Theorem 28. (*Spectral Theorem I.*) Let T be a normal operator on a *complex* vector space V (finite-dimensional). Then V admits an orthonormal basis of eigenvectors for T. Equivalently, we may write:

$$T = \bigoplus_{i=1}^{r} \lambda_i P_i,$$

where $\lambda_i \in \mathbb{C}$ are the eigenvalues of T and $P_i : V \to E(\lambda_i)$ is orthogonal projection on the eigenspace. In particular, $P_i P_j = 0$ if $i \neq j$.

Theorem 29. (*Spectral Theorem II.*) Let T be a self-adjoint operator on a *real* vector space V (finite-dimensional). Then the same conclusion as in Theorem 28 holds for T (with $\lambda_i \in \mathbb{R}$).

Remark. In particular, in this situation, given a function $f : D(f) \to \mathbb{R}$ whose domain $D(f) \subset \mathbb{R}$ contains the spectrum of T, we can define $f(T)$ as the self-adjoint operator given by the formula:

$$f(T) = \bigoplus_{i=1}^{r} f(\lambda_i) P_i.$$

For example, one may use this to define a 'square root' for any positive operator T (in fact, the unique positive square root).

Definition 30. An *isometry* of V is an operator preserving norms: $||Tv|| = ||v||$, for all v. Equivalently (by the polarization identities), T preserves the inner product of two vectors, or: $T^*T = TT^* = I$. In particular, isometries are normal operators. (Any eigenvalue has modulus one.) The set of isometries of V is a *group* (under the operator product), called the 'orthogonal group' $O(n)$ when $V = \mathbb{R}^n$, the 'unitary group' $U(n)$ when $V = \mathbb{C}^n$.

Theorem 31. (*Polar decomposition*) Any operator T admits a factorization $T = UP$, where U is an isometry and P is positive (in part. self-adjoint.) Necessarily $P = \sqrt{T^*T}$. U is uniquely defined by T if (and only if) T is invertible.

Theorem 32. (*Spectral Theorem III.*) Let T be a *normal* operator on a *real* vector space V. Then V admits an orthonormal basis consisting of eigenvectors of T (for real eigenvalues) and 'eigenpairs' for T (for complex conjugate paris of eigenvalues). Equivalently, T can be expressed (symbolically) as an orthogonal direct sum of projections:

$$T = \bigoplus_{i=1}^{r} \gamma_i P_i \oplus \bigoplus_{j=1}^{s} \Lambda(a_j, b_j) Q_j,$$

where γ_i are real eigenvalues for T, $a_j \pm i b_j$ are complex eigenvalues of the complexification $T_{\mathbb{C}}$, $\Lambda(a_j, b_j)$ is the 2×2 matrix

$$\begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}$$

and Q_j is orthogonal projection on the space of 'eigenpairs' for (a_j, b_j).