HARMONIC FUNCTIONS AND POTENTIALS IN $\mathbb{R}^N$

1. The Laplacian and Green's identities

The Laplacian is the second-order differential operator defined on functions $f \in C^2$ by:

$$\Delta f = div(\nabla f),$$

the divergence of the gradient vector field. In standard euclidean coordinates $(x_1, \dots, x_N)$, it is the trace of the Hessian of f:

$$\Delta f = f_{x_1 x_1} + \dots + f_{x_N x_N}.$$

In polar coordinates $f = f(r, \omega)$, where r is distance to $0 \in \mathbb{R}^N$ and $\omega \in S^{N-1}$, Δ has the expression:

$$\Delta f = f_{rr} + \frac{N-1}{r} f_r + \frac{1}{r^2} \Delta_S f,$$

where Δ_S is a second order differential operator acting only on the coordinates ω. In fact, if we write a generic point $\omega \in S$ as $\omega = (\theta \sin\varphi, \cos\varphi) \in \mathbb{R}^{N-1} \times \mathbb{R}$, where $\varphi \in [0, \pi]$ and $\theta \in S^{N-1} = S'$, the operator Δ_S is given by:

$$\Delta_S f = f_{\varphi\varphi} + (N-2) \frac{\cos\varphi}{\sin\varphi} f_\varphi + \frac{1}{\sin^2\varphi} \Delta_{S'} f,$$

where $\Delta_{S'}$ is a second-order differential operator acting only on the variable θ. In particular, setting for the circle S^1: $\Delta_{S^1} f = f_{\theta\theta}$ (where (r, θ) are standard polar coordinates in $\mathbb{R}^2$), this defines (inductively) Δ_S in all dimensions. For example, for $N = 3$, the operator Δ_S is given by:

$$\Delta_S f = f_{\varphi\varphi} + \frac{\cos\varphi}{\sin\varphi} f_\varphi + \frac{1}{\sin^2\varphi} f_{\theta\theta}.$$

An important tool in the theory of potentials is given by *Green's identities* for the Laplacian, which follow easily from the divergence theorem. Recall that if $D \subset \mathbb{R}^N$ is a smooth bounded domain, with unit outward normal vector n at points of its boundary ∂D, and if X is a smooth vector field in the closed domain $\bar{D} = D \cup \partial D$, we have:

$$\int_D div X dvol = \int_{\partial D} X \cdot n dA,$$

where $dvol = dx_1 \dots dx_N$ is the element of volume in R^N (area if $N = 2$) and dA is the element of area on ∂D (arc length if $N = 2$). In the important special case $D = B_R, \partial D = S_R$ (the N-dimensional ball, resp. $(N-1)$-dimensional sphere centered at the origin), we have the relation (in coordinates $(r, \varphi, \theta) \in \mathbb{R}^+ \times [0, \pi] \times S^{N-2}$, as above):

$$dvol = r^{N-1} dr d\omega, \quad dA = R^{N-1} d\omega, \quad d\omega = (\sin\varphi)^{N-2} d\varphi d\theta,$$

where $d\omega, d\theta$ are the elements of 'area' in S^{N-1}, S^{N-2} (resp.); in particular $d\theta$ is just arc length on the unit circle if $N = 3$.

Specializing the divergence theorem to the case $X = g\nabla f$, where f, g are smooth functions on D, we obtain:

$$\int_D [g\Delta f + \nabla g \cdot \nabla f] dvol = \int_{\partial D} g \frac{\partial f}{\partial n} dA, \tag{G1}$$

where $\partial f/\partial n = \nabla f \cdot n$ is the exterior normal derivative of f at the boundary. This is *Green's first identity.* Interchanging f and g and taking the difference, we obtain *Green's second identity*:

$$\int_D [f\Delta g - g\Delta f]dvol = \int_{\partial D}[f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}]dA. \tag{G2}$$

Setting $f = g$ in (G1) we obtain the important identity:

$$\int_D [f\Delta f + |\nabla f|^2]dvol = \int_{\partial D} f\frac{\partial f}{\partial n}dA. \tag{1.1}$$

Definition 1.1 A function $u : D \to \mathbb{R}^n$ is *harmonic* in D if $\Delta u = 0$.

It is clear from (1.1) that if u is harmonic in D (with $D \subset \mathbb{R}^n$ bounded) and either $u = 0$ on ∂D (Dirichlet boundary conditions) or $\frac{\partial u}{\partial n} = 0$ on ∂D (Neumann boundary conditions), then u must be constant in D (and the constant is zero in the Dirichlet case).

This is a good point to introduce the main boundary value problems of potential theory. A physical motivation arises from electrostatics, where Maxwell's equations for the electric field $\mathbb{E}$ due to a charge distribution in space characterized by the charge density function $\rho : \mathbb{R}^3 \to \mathbb{R}$ are (in appropriate units):

$$div\mathbb{E} = \rho, \quad curl\mathbb{E} = 0.$$

The second equation implies the existence of a 'potential function' $u : \mathbb{R}^3 \to \mathbb{R}$ with the property: $\mathbb{E} = -\nabla u$, and hence $\Delta u = -\rho$. The sign $(-)$ is included so that a positive charge 'falls' from regions of higher potential to regions of lower potential (in particular for a point charge at the origin, $u = -1/4\pi r$ increases from $-\infty$ at the origin to 0 at infinity).

The *interior Dirichlet problem* for a bounded domain D asks for the potential u inside a perfect conductor (zero charge density), given the potential f on the boundary:

$$\Delta u = 0 \text{ in } D, \quad u = f \text{ on } \partial D \quad \text{(Dirichlet)}.$$

The *interior Neumann problem* for a bounded domain D asks for the potential function u inside a perfect conductor D, given the normal component of the electric field ($E_n = -\partial u/\partial n$) at boundary points:

$$\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} = f \text{ on } \partial D \quad \text{(Neumann)}.$$

The 'exterior' Dirichlet and Neumann problems are defined similarly- one wishes to find the potential *outside* of D, assuming there are no charges in the exterior.

Exercise. Show that if u_1, u_2 are solutions of the same interior Dirichlet (resp. interior Neumann) problem for the same f, then $u_1 \equiv u_2$ in D (resp. $u_1 \equiv u_2 + const.$ in D).

2. Potentials in $\mathbb{R}^N$.

In this section we consider 'whole-space' problems. The first observation is that, unlike the one-dimensional case (where solutions of $u_{xx} = 0$ are *linear*, and thus define a two-dimensional space), in $\mathbb{R}^N$ for $N \geq 2$ there is a multitude of non-linear harmonic functions. For instance, denoting by $\mathcal{P}_d^n$ the vector space of homogeneous polynomials in n variables, we may consider the subspace $\mathcal{H}_d^n \subset \mathcal{P}_d^n$ of *homogeneous harmonic polynomials* of degree d in n variables. We have:

$$dim(\mathcal{H}_2^2 = 2, \text{ basis: } \{x^2 - y^2, xy\},$$
$$dim(\mathcal{H}_3^2 = 2, \text{ basis: } \{x^3 - 3x^2y, y^3 - 3xy^2\},$$

and in general $dim(\mathcal{H}_d^2) = 2$, with basis given by the real and imaginary parts of $z^d = (x+iy)^d$. Note that $dim(\mathcal{P}_\lceil^\in) = 2d + 1$.

In three variables, since a general homogeneous polynomial $p \in \mathcal{P}^3_d$ may be written in the form:

$$p(x,y,z) = \sum_{i=0}^{d} p_i(x,y)z^i, \quad p_i \in \mathcal{P}^2_i,$$

we have $dim(\mathcal{P}^3_d) = \sum_{i=0}^{d} dim(\mathcal{P}^2_i) = 1+2+\ldots+(d+1) = (d+1)(d+2)/2$. To find the dimension of the subspace $\mathcal{H}^3_d \subset \mathcal{P}^3_d$, we observe that $\mathcal{H}^3_d$ is the *kernel* of the linear map defined by the Laplacian:

$$\Delta : \mathcal{P}^3_d \to \mathcal{P}^3_{d-2}, \quad d \geq 2.$$

It is not hard to show that this linear map is *onto*, and therefore:

$$dim(\mathcal{H}^3_d) = dim(ker\Delta) = dim(\mathcal{P}^3_d) - dim(\mathcal{P}^3_{d-2}) = (d+1)(d+2)/2 - d(d-1)/2 = 2d+1.$$

With this information, it is not hard to find bases for the $\mathcal{H}^3_d$:

$dim(\mathcal{H}^3_2) = 5$, basis: $\{x^2 - y^2, xy, xz, yz, x^2 - z^2\}$

$dim(\mathcal{H}^3_3) = 7$, basis: $\{(x62 - y^2)z, (x^2 - z^2)y, x^3 - 3xy^2, y^3 - 3x^2y, z^3 - 3x^2z, (y^2 - z^2)x, xyz\}$.

In general, one gets enough examples for a basis of $\mathcal{H}^3_d$ by (i)multiplying elements of $\mathcal{H}^2_{d-1}$ by z; (ii)permuting variables.

It is also natural to look for examples of *rotationally symmetric* harmonic functions in $\mathbb{R}^N$, that is, harmonic functions depending only on distance to the origin, r. A harmonic $u = u(r)$ is a solution of the ordinary differential equation:

$$u_{rr} + \frac{N-1}{r} f_r = 0,$$

which has solutions:

$$u(r) = C_1 \log r + C_2, N = 2;$$
$$u(r) = C_1 r^{2-N} + C_2, N \geq 3.$$

Thus we see that, except for constants, there are no rotationally symmetric harmonic functions defined on all of $\mathbb{R}^N$ (only one $\mathbb{R}^N - \{0\}$).

Shifting the origin to an arbitary $x_0 \in \mathbb{R}^N$ and choosing particular values for the constants C_1, C_2, we obtain an important definition:

Definition 1.2. The *Green's function* for $\mathbb{R}^N$ with 'pole' at $x_0 \in \mathbb{R}^n$ is:

$$G_{x_0}(x) = \frac{1}{(N-2)\omega_{N-2}||x - x_0||^{N-2}}, \quad (N \geq 3); G_{x_0}(x) = -\frac{1}{2\pi} \log ||x - x_0||, \quad (N = 2).$$

Note that the Green's function is positive and decays to zero at infinity for $N \geq 3$, but changes sign and does not decay at infinity if $N = 2$ (this is reflected in vastly different qualitative properties for Brownian motion when for $N = 2$ and $N \geq 3$). When $N = 3$, $G_{x_0}(x) = \frac{1}{4\pi||x-x_0||}$ has the physical interpretation 'electric potential produced by a unit point positive charge at x_0'. It solves the equation:

$$\Delta G_{x_0} = -\delta_{x_0},$$

(the 'delta function at x_0', and corresponds to the electic field:

$$\mathbb{E}(x) = -\nabla G_{x_0}(x) = \frac{1}{4\pi||x - x_0||^2}.$$