EVOLUTION OF THE GEOMETRY UNDER MCF.

Evolution of the unit normal. Regarding N as a function on M_0: $N(t) : M_0 \to \mathbb{R}^{n+1}$, we compute in a local orthonormal frame (e_i) on M_0, using $\partial_t F = HN$:

$$\partial_t N = g^{ij}\langle \partial_t N, dFe_i\rangle dFe_j = -g^{ij}\langle N, \partial_t(dFe_i)\rangle dFe_j$$
$$= -g^{ij}(dHe_i)dFe_j = -\nabla^t H,$$

the gradient vector field of H in the induced metric g_t.

For a hypersurface $M \subset \mathbb{R}^{n+1}$ with unit normal vector field $N : M \to S^n$ (regarded as a vector-valued function on M), picking the frame (e_i) on M so that $\nabla_{e_i} e_j = 0$ at a fixed $p \in M$, we compute at p:

$$e_i(N) = -A(e_i, e_j)e_j,$$

and using the Codazzi equation:

$$e_k(e_i(N)) = -\nabla_{e_k} A(e_i, e_j)e_j - A(e_i, e_j)e_k(e_j)$$
$$= -(\nabla_{e_j} A)(e_k, e_i)e_j - A(e_i, e_j)A(e_j, e_k)N,$$

giving for the Hessian of N:

$$\nabla^2_{e_k,e_i} N = -\nabla^M(A(e_k, e_i)) - A^2(e_k, e_i)N,$$

and for the Laplacian of N:

$$\Delta_M N = -\nabla^M H - |A|^2 N.$$

Combining these results, we have for the evolution of the unit normal under mean curvature flow the 'heat equation':

$$\partial_t N - \Delta_t N = |A|^2 N.$$

Remark- to be expanded. The operator $\Delta_g f + |A|^2 f$ also occurs in the formula for *second variation of area.* Assume $F_\epsilon : M_0 \to \mathbb{R}^{n+1}$ is a family of immersions, constant equal to F_0 outside a compact subset $D \subset M_0$. Suppose, in addition (i)F_0 is minimal, i.e. has zero mean curvature; (ii) the variational vector field $V = \frac{d}{d\epsilon}|_{\epsilon=0} F_\epsilon$ is normal (with support contained in D). Then we have for the second derivative of the area of M_ϵ in D:

$$\frac{d^2}{d\epsilon^2}_{|\epsilon=0} Area_D(M_\epsilon) = -\int_{M_0} \langle \Delta_0 V + |A|^2 V, V\rangle \omega_0.$$

Time derivative of the second fundamental form. To understand precisely what this means, recall we have a path of embeddings $F_t : M_0 \to M_t \subset \mathbb{R}^{n+1}$. We define the quadratic form $\partial_t A$ on M_t using the push-forward of vector fields:

$$\partial_t A(X^{F_t}, Y^{F_t}) = \frac{d}{dt} A(dF_t X, dF_t Y),$$

for tangent vector fields X, Y on M_0. Denote by ∇^t the pullback connection (by F_t) on M_0 and recall the Hessian formulas ($F = F_t$ henceforth):

$$(\nabla^t)^2 F(e_i, e_j) = A(dFe_i, dFe_j)N, \quad e_i \in TM_0$$

(so the Hessian of F_t is purely normal) and (for any function f on M_0):

$$(\nabla^t)^2 f(e_i, e_j) - (\nabla^0)^2 f(e_i, e_j) = -df(\Gamma(e_i, e_j)), \quad \Gamma(X, Y) := \nabla^t_X Y - \nabla^0_X Y.$$

From this it follows that if Z is a tangent vector field on $M_t \subset \mathbb{R}^{n+1}$:

$$\langle (\nabla^0)^2 F(e_i, e_j), Z\rangle = \langle dF(\Gamma(e_i, e_j)), Z\rangle$$

(on the left, F is regarded as a vector-valued function on M_0). Thus we can express the function to be differentiated in terms of a fixed connection on M_0:

$$A(dFe_i, dFe_j) = \langle (\nabla^0)^2 F(e_i, e_j), N\rangle.$$

Using $\partial_t F = HN$ and $\partial_t N = -\nabla^{M_t} H$, we obtain:

$$\frac{d}{dt} A(dFe_i, dFe_j) = \langle (\nabla^0)^2 (HN)(e_i, e_j), N\rangle - \langle (\nabla^0)^2 F(e_i, e_j), \nabla^{M_t} H\rangle.$$

One easily computes that:

$$\langle (\nabla^0)^2 (HN)(e_i, e_j), N\rangle = (\nabla^0)^2 H(e_i, e_j) + H\langle (\nabla^0)^2 N(e_i, e_j), N\rangle,$$

and earlier we found for the Hessian of N as a function on M_t:

$$\langle (\nabla^t)^2 N(dFe_i, dFe_j), N\rangle = -A^2(dFe_i, dFe_j).$$

The observation above applied to $Z = \nabla^{M_t} H$ gives:

$$\langle (\nabla^0)^2 F(e_i, e_j), \nabla^{M_t} H\rangle = \langle dF(\Gamma(e_i, e_j)), \nabla^{M_t} H\rangle.$$

Thus we find:

$$\frac{d}{dt} A(dFe_i, dFe_j) = (\nabla^0)^2 H(e_i, e_j) - \langle dF(\Gamma(e_i, e_j)), \nabla^{M_t} H\rangle - HA^2(dFe_i, dFe_j)$$

$$= (\nabla^t)^2 H(dFe_i, dFe_j) - HA^2(dFe_i, dFe_j).$$

(For the last equality we used the fact that:

$$\langle dF(\Gamma(e_i, e_j)), \nabla^{M_t} H\rangle = dH(\Gamma(e_i, e_j)),$$

regarding H as a function on M_0 on the right-hand side). On the other hand, when writing the Hessian $(\nabla^t)^2 H$ we think of H as a function on M_t. We conclude:

$$\partial_t A(Z, W) = (\nabla^t)^2 H(Z, W) - HA^2(Z, W),$$

as quadratic forms on M_t.

Evolution of the metric. A similar definition and calculation applies to the metric tensor:

$$\partial_t g(X^F, Y^F) = \frac{d}{dt}\langle dF_t X, dF_t Y\rangle = -2HA(X^F, Y^F).$$

Remark- to be expanded. Using the Gauss equations, one finds easily the following relation:

$$Rc = HA - A^2,$$

where Rc is the Ricci tensor:

$$Rc(e_i, e_j) = \langle R(e_i, e_l)e_l, e_j\rangle.$$

Thus the evolution of the metric can be written in the form:

$$\partial_t g = -2Rc - 2A^2,$$

to be compared with Ricci flow:

$$\partial_t g = -2Rc.$$

Is there more than a merely formal connection? Can the difference be eliminated by a diffeomorphism? What happens in higher codimension?

Simons' identity. This is a 'Bochner-type' formula relating the Laplacian of the second fundamental form and the Hessian of mean curvature:

$$\Delta A = \nabla^2 H - |A|^2 A + HA^2,$$

where A^2 is the symmetric two-tensor defined (for codimension 1 submanifolds) by:

$$A^2(X,Y) = \sum_i A(X,e_i)A(e_i,Y)$$

((e_i) an arbitrary orthonormal frame).

Recall we define the curvature tensor by:

$$R_{X,Y}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

The curvature tensor can be used to commute covariant derivatives of an arbitrary (2,0) tensor:

Lemma. If T is a 2-tensor on M:

$$(\nabla^2_{X,Y}T - \nabla^2_{Y,X}T)(Z,W) = -T(R_{X,Y}Z,W) - T(Z,R_{X,Y}W).$$

Recall that, by definition:

$$(\nabla^2_{X,Y}T) = \nabla_X((\nabla_Y T)) - \nabla_{\nabla_X Y}T$$

and:

$$(\nabla_X T)(V,W) = X(T(V,W)) - T(\nabla_X V, W) - T(V, \nabla_X W).$$

To prove the lemma, given $p \in M$ extend $X(p), Y(p), Z(p), W(p)$ locally so that all covariant derivatives vanish at p. Then at p we have:

$$(\nabla^2_{X,Y}T)(Z,W) = X(Y(T(Z,W))) - T(\nabla_X \nabla_Y Z, W) - T(Z, \nabla_X \nabla_Y W).$$

Now interchange X and Y and take the difference to obtain the result (note that, at p, $[X,Y] = \nabla_X Y - \nabla_Y X = 0$.)

We also need the Gauss and Codazzi equations for a codimension 1 submanifold of $\mathbb{R}^{n+1}$. The Gauss equation expresses the (4,0) curvature tensor of M in terms of the second fundamental form:

$$\langle R_{X,Y}Z, W\rangle = -[A(X,Z)A(Y,W) - A(Y,Z)A(X,W)].$$

The Codazzi equation states that the covariant derivative of A, the (3,0) tensor:

$$(\nabla_X A)(Y,Z)$$

is symmetric in all three arguments X, Y, Z.

Proof of Simons' identity. Fix $p \in M$ and let (e_i) be a local orthonormal frame so that $\nabla_{e_i} e_j(p) = 0$ for all i, j. (Notation: $h_{ij} = A(e_i, e_j)$, $R_{ijkl} = \langle R_{e_i,e_j} e_k, e_l \rangle$).

Computing at p, using the Codazzi identity (and summation convention):

$$\nabla^2_{e_k,e_k}(e_i, e_j) = e_k((\nabla_{e_k} A)(e_i, e_j)) = e_k((\nabla_{e_i} A)(e_k, e_j)) = (\nabla^2_{e_k,e_i} A)(e_k, e_j)$$

and now using the lemma:

$$= (\nabla^2_{e_i,e_k} A)(e_k, e_j) - A(R_{e_k,e_i} e_k, e_j) - A(e_k, R_{e_k,e_i} e_j)$$

$$= \nabla_{e_i}((\nabla_{e_k} A)(e_k, e_j)) - R_{kikl} h_{lj} - R_{kijl} h_{kl}.$$

Using Codazzi again, combined with the Gauss equation:

$$= \nabla_{e_i}((\nabla_{e_j} A)(e_k, e_k)) + (h_{kk} h_{il} - h_{ik} h_{il}) h_{lj} + (h_{kj} h_{il} - h_{ij} h_{kl}) h_{kl}$$

$$= (\nabla^2_{e_i,e_j} A)(e_k, e_k) + h_{kk} h_{il} h_{lj} - h_{ij} (h_{kl})^2$$

$$= \nabla^2_{e_i,e_j} H + H A^2(e_i, e_j) - |A|^2 A(e_i, e_j).$$

Heat equation for the second fundamental form. For a hypersurface $(M_t)_{t \in [0,T)}$ evolving under mean curvature flow, combining the expression obtained above for $\partial_t A$ and Simons' identity for $\Delta_t A$, we find:

$$\partial_t A - \Delta_t A = -2HA^2 + |A|^2 A,$$

as quadratic forms on M_t.

We can think of the equations just derived as a system of coupled evolution equations for the pair (g, A), regarded as symmetric 2-tensors on M_0:

$$\partial_t g = -2(tr_g A) A, \quad \partial_t A = \Delta_g A - 2(tr_g A) A^2 + |A|^2_g A$$

or, in local coordinates:

$$\partial_t g_{ij} = -2 g^{kl} h_{kl} h_{ij},$$

$$\partial_t h_{ij} = g^{kl} h_{ij;k;l} - 2 g^{kl} g^{mn} h_{kl} h_{im} h_{nj} + g^{kl} g^{mn} h_{km} h_{nl} h_{ij}.$$

Question: Does this system preserve the integrability conditions ('constraints') expressed by the Gauss and Codazzi equations? (It should.) What are its gauge-invariance properties? Can one show local existence? (It is not obviously parabolic, but should be.)

Remark: The following is a classical theorem: let (M_0, g) be a *connected and simply-connected* Riemannian n-manifold, endowed with a symmetric 2-tensor A. Assume g and A satisfy the Gauss and Codazzi equations. Then there exists an *isometric immersion* F of M_0 into $\mathbb{R}^{n+1}$ and a field N of unit normal vectors on $F(M_0)$ so that the second fundamental form defined by N is A. Any two such immersions F differ by an isometry of $\mathbb{R}^{n+1}$[Kobayashi-Nomizu, vol.2 p.52, Thm 7.2].